FOLIATED GROUPOIDS AND THEIR INFINITESIMAL DATA

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Abstract. In this work, we study Lie groupoids equipped with multiplicative foliations and the corresponding infinitesimal data. We determine the infinitesimal counterpart of a multiplicative foliation in terms of its core and sides together with a partial connection satisfying special properties, giving rise to the concept of IM-foliation on a Lie algebroid. The main result of this paper shows that if G is a source simply connected Lie groupoid with Lie algebroid A, then there exists a one-to-one correspondence between multiplicative foliations on G and IM-foliations on the Lie algebroid A.

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1. Introduction

This paper is the first part of a series of two articles devoted to the study of the infinitesimal data of Dirac groupoids, i.e. Lie groupoids equipped with Dirac structures suitably compatible with the groupoid multiplication.

Since Dirac structures generalize Poisson bivectors, closed 2-forms and regular foliations, multiplicative Dirac structures [26, 13] provide a unified framework for the study of a variety of geometric structures compatible with a group(oid) multiplication, including Poisson groupoids and multiplicative closed 2-forms. It is well known that Poisson groupoids are infinitesimally described by Lie bialgebroids [23]. Similarly, the infinitesimal counterpart of multiplicative closed 2-forms on a Lie groupoid are IM-2-forms on the corresponding Lie algebroid [6, 4]. Here we are concerned with Lie groupoids equipped with a multiplicative foliation. In particular, our main goal is to describe multiplicative foliations at the infinitesimal level.

Consider a Lie group G with Lie algebra $\mathfrak{g}$ and multiplication map $\mathsf{m} : G \times G \to G$. Then the tangent space TG of G is also a Lie group with multiplication map $T\mathsf{m} : TG \times TG \to TG$. It was observed in [15, 14, 25] that a multiplicative distribution $F_G \subseteq TG$, that is, a distribution on G that is a subgroup of TG, is the bi-invariant image of an ideal $\mathfrak{f} \subseteq \mathfrak{g}$. Hence, it is automatically an *involutive subbundle* of TG which is *completely determined by its fiber* $\mathfrak{f}$ *over the unit* of the Lie group.

If $G \rightrightarrows M$ is a Lie groupoid, then the application of the tangent functor to all the groupoid maps gives rise to a Lie groupoid $TG \rightrightarrows TM$, called the tangent groupoid. A distribution $F_G \subseteq TG$ on G is said to be **multiplicative** if F_G is a subgroupoid of $TG \rightrightarrows TM$ over $F_M \subseteq TM$. In this more general situation it is no longer true that F_G is automatically involutive and has constant rank. Indeed, every smooth manifold can be viewed as a Lie groupoid over itself, and, in this case, any distribution $F \subseteq TM$ is multiplicative.

We study in this paper how the easy properties of multiplicative distributions on Lie groups can be extended to the more general framework of Lie groupoids. A **multiplicative foliation** on a Lie groupoid $G \rightrightarrows M$ is a multiplicative *subbundle* $F_G \subseteq TG$ which is also involutive with respect to the Lie bracket of vector fields. We say that the pair $(G \rightrightarrows M, F_G)$ is a **foliated groupoid**. The main goal of this paper is to describe foliated groupoids infinitesimally, that is, in terms of Lie algebroid data.

First, we observe that given a Lie groupoid $G \rightrightarrows M$ with Lie algebroid A, then the space of units $F_M \subseteq TM$ of a multiplicative foliation $F_G \subseteq TG$ is necessarily an involutive subbundle, and similarly the core $F_{\mathsf{c}} := F_G \cap T^{\mathsf{s}}_M G$ of F_G is a Lie subalgebroid of the Lie algebroid $A = T^{\mathsf{s}}_M G$. In particular, a multiplicative foliation $F_G \subseteq TG$ can be thought of as a $\mathcal{LA}$-subgroupoid of $TG \rightrightarrows TM$. As a consequence, following [20, 26] we prove that the Lie algebroid of F_G is, modulo the canonical identification $A(TG) \simeq TA$, a subalgebroid $F_A \longrightarrow F_M$ of the tangent Lie algebroid $TA \longrightarrow TM$ which at the same time is an involutive subbundle $F_A \subseteq TA$. In this case we say that (A, F_A) is a **foliated algebroid** and F_A is referred to as a **morphic foliation** on A. We show that if $G \rightrightarrows M$ is a source simply connected Lie groupoid with Lie algebroid A, then there is a one-to-one correspondence between multiplicative foliations on G and morphic foliations on A.

We prove that if G is a Lie groupoid with Lie algebroid A endowed with a multiplicative foliation F_G with core $F_{\mathsf{c}} \subseteq A$ and units $F_M \subseteq TM$, then there is a well-defined partial F_M-connection ∇ on A/F_{c}, which is naturally induced by the Bott connection associated to the foliation F_G on G. We show that this connection is flat and allows us to compute explicitly the Lie algebroid of $F_G \rightrightarrows F_M$ in terms of restrictions to $F_M \subseteq TM$ of special sections of the Lie algebroid $A(TG) \longrightarrow TM$ of the tangent groupoid. Furthermore, we show that the involutivity of the multiplicative subbundle $F_G \subseteq TG$ is completely encoded in some data involving only F_M, F_{c} and the flatness of the connection ∇.

Given a foliated algebroid (A, F_A), we show that the core F_{c} and side F_M of F_A are Lie subalgebroids of A and TM, respectively. We prove that, here also, there is a flat partial F_M-connection on A/F_{c} which is naturally induced by the Bott connection defined by the foliation F_A on A. In addition, we show that if A integrates to a Lie groupoid G and $F_A \subseteq TA$ integrates to a multiplicative foliation $F_G \subseteq TG$, then the partial connections induced by F_A and F_G coincide. We say that the data $(A, F_M, F_{\mathsf{c}}, \nabla)$ is an **IM-foliation** on A.

The main result of this work states that the data $(A, F_M, F_{\mathsf{c}}, \nabla)$ determines completely the foliated algebroid (A, F_A) in the sense that F_A can be reconstructed out of $(A, F_M, F_{\mathsf{c}}, \nabla)$. In particular, if $G \rightrightarrows M$ is a source simply connected Lie groupoid with Lie algebroid A, then there is a one-to-one correspondence between multiplicative foliations on G and IM-foliations $(A, F_M, F_{\mathsf{c}}, \nabla)$ on A.

The partial connection induced by a multiplicative foliation already appeared in [15], where the existence of its parallel sections is used to show that under some completeness conditions, there is a groupoid structure on the leaf space $G/F_G \rightrightarrows M/F_M$ of the multiplicative foliation. Analogously, we study the leaf space of a foliated algebroid (A, F_A). We prove that, whenever the leaf space M/F_M is a smooth manifold and the partial connection has trivial holonomy, then there is a natural Lie algebroid structure on $A/F_A \longrightarrow M/F_M$. We also show that if a foliated groupoid $(G \rightrightarrows M, F_G)$ is such that the completeness conditions of [15] are fulfilled, then the Lie groupoid $G/F_G \rightrightarrows M/F_M$ has Lie algebroid $A/F_A \longrightarrow M/F_M$. Several examples of foliated groupoids and algebroids are discussed, including actions by groupoid and algebroid automorphisms, complex Lie groupoids and complex Lie algebroids and integrable regular Dirac structures versus regular presymplectic groupoids[1].

Given a Lie groupoid $G \rightrightarrows M$ with Lie algebroid A, it was proved in [7] that the cotangent bundle T^*G has a Lie groupoid structure over A^*. In particular, the direct sum vector bundle $TG \oplus T^*G$ inherits a Lie groupoid structure over $TM \oplus A^*$. It is easy to see that if $F_G \subseteq TG$ is a multiplicative foliation on G, then its annihilator $F_G^\circ \subseteq T^*G$ is a Lie subgroupoid. Hence, the subbundle $F_G \oplus F_G^\circ \subseteq TG \oplus T^*G$ is an example of a multiplicative Dirac structure on G (see [12, 26, 13]). Every multiplicative Dirac structure induces Lie algebroid structures on the core and the base [12]. Furthermore there is a Courant algebroid that is canonically associated to the Dirac groupoid [12], which generalizes the Lie bialgebroid of a Poisson groupoid and is closely related to the partial connections studied in this paper.

The understanding of multiplicative Dirac structures at the infinitesimal level in the spirit of this paper is a very interesting goal, because this generalizes simultaneously the correspondences between Poisson groupoids and Lie bialgebroids [23], multiplicative closed 2-forms and IM-2-forms [6, 4], and the results in this paper. This infinitesimal description of Dirac groupoids can be made with similar techniques as here, by using the core and side algebroids of a multiplicative Dirac structure D_G on G together with the Courant algebroid associated to D_G. We will treat this problem in a separate paper [9].

Remark. Our result on the correspondence between IM-foliations and foliated algebroids (foliated groupoids) was already stated in [11], but we were not aware of that while working on this subject.

Structure of the paper and main results. In Section 2, we start by recalling background on Lie groupoid theory, especially the tangent and cotangent groupoids associated to a Lie groupoid, as well as the Lie algebroid structures on $A(TG) \to TM$ and on $TA \to TM$ in terms of linear and core sections. We also discuss briefly the definition and some properties of flat partial connections.

[1] Here, *regular* means that the characteristic distributions have constant rank.

In Section 3, we define foliated Lie groupoids, together with the associated subalgebroids $F_M \subseteq TM$ and $F_{\mathsf{c}} \subseteq A$. Our first main results are summarized in the following theorem.

Theorem 1 *Let $G \rightrightarrows M$ be a Lie groupoid endowed with a multiplicative subbundle $F_G \subseteq TG$. Then there is a map*

$$\tilde{\nabla} : \Gamma(F_M) \times \Gamma(A) \to \Gamma(A/F_{\mathsf{c}})$$

such that the following holds: F_G is involutive if and only if

a) *$F_M \subseteq TM$ is involutive,*
b) *for any $\bar{X} \in \Gamma(F_M)$, $\tilde{\nabla}(\bar{X}, \cdot)$ vanishes on sections of F_{c} and*
c) *the induced map*

$$\nabla : \Gamma(F_M) \times \Gamma(A/F_{\mathsf{c}}) \to \Gamma(A/F_{\mathsf{c}}).$$

is then a flat partial F_M-connection on A/F_{c}.

The connection ∇ has in this case the following properties:

(1) *If $a \in \Gamma(A)$ is ∇-parallel, i.e., $\nabla_{\bar{X}}\bar{a} = 0$ for all $\bar{X} \in \Gamma(F_M)$, where $\bar{a}$ is the class of a in $\Gamma(A/F_{\mathsf{c}})$, then $[a, b] \in \Gamma(F_{\mathsf{c}})$ for all $b \in \Gamma(F_{\mathsf{c}})$.*
(2) *If $a, b \in \Gamma(A)$ are ∇-parallel, then $[a, b]$ is also ∇-parallel.*
(3) *If $a \in \Gamma(A)$ is ∇-parallel, then $[\rho(a), \bar{X}] \in \Gamma(F_M)$ for all $\bar{X} \in \Gamma(F_M)$. That is, $\rho(a)$ is ∇^{F_M}-parallel, where ∇^{F_M} denotes the Bott connection defined by F_M.*

Furthermore, $a \in \Gamma(A)$ is ∇-parallel if and only if the right invariant vector field a^r on G is parallel with respect to the Bott connection ∇^{F_G} defined by F_G. That is

$$[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G).$$

In Section 4, we introduce and study the concept of foliated Lie algebroids. We compute explicitly the Lie algebroid of a multiplicative foliation in terms of parallel sections of the flat partial connection of Theorem 1, and we show the following integration result.

Theorem 2 *Let $(G \rightrightarrows M, F_G)$ be a foliated groupoid. Then $(A, F_A = j_G^{-1}(A(F_G)))$ is a foliated algebroid.*[2]

Conversely, let (A, F_A) be a foliated Lie algebroid. Assume that A integrates to a source simply connected Lie groupoid $G \rightrightarrows M$. Then there is a unique multiplicative foliation F_G on G such that $F_A = j_G^{-1}(A(F_G))$.

A foliated algebroid (A, F_A) induces similar data $(A, F_M, F_{\mathsf{c}}, \nabla)$ as does a foliated groupoid.

Theorem 3 *Let (A, F_A) be a foliated algebroid. Then $F_M \subseteq TM$ and $F_{\mathsf{c}} \subseteq A$ are subalgebroids and there is a natural partial F_M-connection ∇^A on A/F^c induced by F_A. This connection has the same properties as the connection found in Theorem 1.*

[2] Here, $j_G : TA \to A(TG)$ is the canonical identification.

The parallel sections of ∇^A *are exactly the sections* $a \in \Gamma(A)$ *such that* $[a^\uparrow, \Gamma_A(F_A)] \subseteq \Gamma_A(F_A)$, *that is the core vector field* $a^\uparrow$ *on* A *is parallel with respect to the Bott connection defined by* F_A.

The data induced by a foliated groupoid and its foliated algebroid are the same, according to the next result.

Theorem 4 *Assume that* A *integrates to a Lie groupoid* $G \rightrightarrows M$ *and that* $j_G(F_A)$ *integrates to* $F_G \subseteq TG$. *Then* $\nabla = \nabla^A$.

The next theorem, which is proved in Section 5, is the main result of this work. We prove that the data $(A, F_M, F_{\mathsf{c}}, \nabla)$, which is called an **IM-foliation** on A, is what should be considered as the infinitesimal data of a multiplicative foliation.

Theorem 5 *Let* $(A \to M, \rho, [\cdot\,,\cdot])$ *be a Lie algebroid and* $(A, F_M, F_{\mathsf{c}}, \nabla)$ *an IM-foliation on* A. *Then there exists a unique morphic foliation* F_A *on* A *with side* F_M *and core* F_{c} *such that the induced connection* ∇^A *is equal to* ∇.

If A *integrates to a source simply connected Lie groupoid* $G \rightrightarrows M$, *there is a unique multiplicative foliation* F_G *on* G *with associated infinitesimal data* $(A, F_M, F_{\mathsf{c}}, \nabla)$.

In Section 6 we study the leaf space of a foliated algebroid, and relate it in the integrable case to the leaf space of the corresponding foliated groupoid [15].

Theorem 6 *Let* $(A, F_M, F_{\mathsf{c}}, \nabla)$ *be an IM-foliation on* A *and assume that the leaf space* M/F_M *is a smooth manifold and* ∇ *has trivial holonomy. Then there is natural Lie algebroid structure on* A/F_A *over* M/F_M *such that the projection* (π, π_M) *is a Lie algebroid morphism.*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "q_A"'] & A/F_A \arrow[d, "{[q_A]}"] \\ M \arrow[r, "\pi_M"'] & M/F_M \end{tikzcd}$$

If (A, F_A) *integrates to* $(G \rightrightarrows M, F_G)$ *and* $G/F_G \rightrightarrows M/F_M$ *is a Lie groupoid, then* A/F_A *with the structure above is the Lie algebroid of* $G/F_G \rightrightarrows M/F_M$.

We show in the last section that regular Dirac manifolds provide an interesting example of IM-foliations. The corresponding multiplicative foliation is the kernel of the presymplectic groupoid integrating the Dirac structure. Finally, we discuss shortly the relation of this work with the foliated algebroids in the sense of Vaisman [33].

Notation. Let M be a smooth manifold. We will denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the spaces of (local) smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \to M$, the space of (local) sections of E will be written $\Gamma(E)$. We will write $\mathrm{Dom}(\sigma)$ for the open subset of the smooth manifold M where the local section $\sigma \in \Gamma(E)$ is defined.

The flow of a vector field X will be written $\phi^X_\cdot$, unless specified otherwise.

Let $f : M \to N$ be a smooth map between two smooth manifolds M and N. Then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be f**-related** if $Tf \circ X = Y \circ f$ on $\mathrm{Dom}(X) \cap f^{-1}(\mathrm{Dom}(Y))$. We write then $X \sim_f Y$.

The pullback or restriction of a vector bundle $E \to M$ to an embedded submanifold N of M will be written $E|_N$. In the special case of the tangent and cotangent spaces of M, we will write $T_N M$ and $T^*_N M$. If $f : M \to N$ is a smooth surjective submersion, we write $T^f M$ for the kernel of $Tf : TM \to TN$.

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2. Background

2.1. **Basics on Lie Theory.**

2.1.1. *Lie groupoids.* A **groupoid** G over the **units** M will be written $G \rightrightarrows M$. The **source** and **target** maps are denoted by $\mathsf{s}, \mathsf{t} : G \longrightarrow M$ respectively, the **unit section** $\epsilon : M \longrightarrow G$, the inversion map $\mathsf{i} : G \longrightarrow G$ and the multiplication $\mathsf{m} : G_{(2)} \longrightarrow G$, where $G_{(2)} = \{(g, h) \in G \times G \mid \mathsf{t}(h) = \mathsf{s}(g)\}$ is the set of composable groupoid pairs.

A groupoid G over M is called a **Lie groupoid** if both G and M are smooth Hausdorff manifolds, the source and target maps $\mathsf{s}, \mathsf{t} : G \longrightarrow M$ are surjective submersions, and all the other structural maps are smooth. Throughout this work we only consider Lie groupoids.

Let G and G' be Lie groupoids over M and M', respectively. A **morphism** of Lie groupoids is a smooth map $\Phi : G' \longrightarrow G$ over $\phi : M' \to M$ that is compatible with all the structural maps. When Φ is injective we say that G' is a **Lie subgroupoid** of G. See [21] for more details.

2.1.2. *Lie algebroids.* A **Lie algebroid** is a vector bundle $q_A : A \longrightarrow M$ equipped with a Lie bracket $[\cdot,\cdot]_A$ on the space of smooth sections $\Gamma_M(A)$ and a vector bundle map $\rho_A : A \longrightarrow TM$ called the **anchor**, such that

$$[a, fb]_A = f[a, b]_A + (\pounds_{\rho_A(a)} f)b,$$

for every $a, b \in \Gamma_M(A)$ and $f \in C^\infty(M)$.

Let A and A' be Lie algebroids over M and M', respectively. A **Lie algebroid morphism** is a bundle map $\phi : A' \longrightarrow A$ over $\phi : M' \to M$ which is compatible with the anchor and bracket, see [21] for more details. If $\phi : A' \longrightarrow A$ is injective, we say that A' is a **Lie subalgebroid** of A.

2.1.3. *The Lie functor.* Let $G \rightrightarrows M$ be a Lie groupoid. The Lie algebroid of G is defined in this paper to be $AG = T^{\mathsf{s}}_M G$, with anchor $\rho_{AG} := T\mathsf{t}|_{AG}$ and bracket $[\cdot\,,\cdot]_{AG}$ defined by using right invariant vector fields.

We will write $A(\cdot)$ for the functor that sends Lie groupoids to Lie algebroids and Lie groupoid morphisms to Lie algebroid morphisms. *For simplicity,* $(AG, \rho_{AG}, [\cdot\,,\cdot]_{AG})$ *will be written* $(A, \rho, [\cdot\,,\cdot])$ *in the following.*

Note that if $a \in \Gamma_M(A)$, the vector field a^r satisfies $a^r \sim_{\mathsf{t}} \rho(a) \in \mathfrak{X}(M)$ since we have $T_g\mathsf{t} a^r(g) = T_g\mathsf{t}(T_{\mathsf{t}(g)} r_g\, a(\mathsf{t}(g))) = T_{\mathsf{t}(g)}\mathsf{t}\, a(\mathsf{t}(g))$ for all $g \in G$.

2.1.4. *Tangent and cotangent groupoids.* Let G be a Lie groupoid over M with Lie algebroid A. The tangent bundle TG has a natural Lie groupoid structure over TM. This structure is obtained by applying the tangent functor to each of the structure maps defining G (source, target, multiplication, inversion and identity section). We refer to TG with the groupoid structure over TM as the **tangent groupoid** of G [21]. The set of composable pairs $(TG)_{(2)}$ of this groupoid is equal to $T(G_{(2)})$. For $(g,h) \in G_{(2)}$ and a pair $(v_g, w_h) \in (TG)_{(2)}$, the multiplication is

$$v_g \star w_h := T\mathsf{m}(v_g, w_h).$$

As in [20], we define **star vector fields on** G or **star sections of** TG to be vector fields $X \in \mathfrak{X}(G)$ such that there exists $\bar{X} \in \mathfrak{X}(M)$ with $X \sim_{\mathsf{s}} \bar{X}$ and $\bar{X} \sim_{\epsilon} X$, i.e., X and $\bar{X}$ are s-related and $\bar{X}$ and X are ϵ-related, or X restricts to $\bar{X}$ on M. We write then $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$. In the same manner, we can define t-star sections, $X \overset{\star}{\sim}_{\mathsf{t}} \bar{X}$ with $\bar{X} \in \mathfrak{X}(M)$ and $X \in \mathfrak{X}(G)$. It is easy to see that the tangent space TG is spanned by star vector fields at each point in $G \setminus M$. Not also that the Lie bracket of two star sections of TG is easily seen to be a star section.

Consider now the cotangent bundle T^*G. It was shown in [7], that T^*G is a Lie groupoid over A^*. The source and target of $\alpha_g \in T^*_gG$ are defined by

$$\tilde{\mathsf{s}}(\alpha_g) \in A^*_{\mathsf{s}(g)}, \qquad \tilde{\mathsf{s}}(\alpha_g)(a) = \alpha_g(Tl_g(a - T\mathsf{t}(a))) \quad \text{for all} \quad a \in A_{\mathsf{s}(g)}$$

and

$$\tilde{\mathsf{t}}(\alpha_g) \in A^*_{\mathsf{t}(g)}, \qquad \tilde{\mathsf{t}}(\alpha_g)(b) = \alpha_g(Tr_g(b)) \quad \text{for all} \quad b \in A_{\mathsf{t}(g)}.$$

The multiplication on T^*G will also be denoted by $\star$ and is defined by

$$(\alpha_g \star \beta_h)(v_g \star w_h) = \alpha_g(v_g) + \beta_h(w_h)$$

for $(v_g, w_h) \in T_{(g,h)}G_{(2)}$.

We refer to T^*G with the groupoid structure over A^* as the **cotangent groupoid** of G.

2.2. **The tangent algebroid.** Let M be a smooth manifold. The tangent bundle of M is denoted by $p_M : TM \longrightarrow M$. Consider now $q_A : A \longrightarrow M$ a vector bundle over M. The tangent bundle TA has a natural structure of vector bundle over TM, defined by applying the tangent functor to each of the structure maps that define the vector bundle $q_A : A \to M$. This yields a commutative diagram

$$\begin{array}{ccc} TA & \xrightarrow{Tq_A} & TM \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p_M} \\ A & \xrightarrow[q_A]{} & M \end{array} \tag{2.1}$$

where all the structure maps of $TA \to TM$ are vector bundle morphisms over the corresponding structure maps of $A \to M$. In the terminology of [21, 27] this defines a double vector bundle. Note finally that the zero element in the fiber of TA over $v \in TM$ is $T0^A v$, i.e., the zero section is just $T0^A \in \Gamma_{TM}(TA)$.

2.2.1. *The tangent algebroid* $TA \to TM$. Assume now that $q_A : A \longrightarrow M$ has a Lie algebroid structure with anchor map $\rho : A \longrightarrow TM$ and Lie bracket $[\cdot,\cdot]$ on $\Gamma_M(A)$. Then there is a Lie algebroid structure on TA over TM referred to as the **tangent Lie algebroid**.

In order to describe explicitly the algebroid structure on $Tq_A : TA \longrightarrow TM$, we recall first that there exists a **canonical involution**

$$
\begin{tikzcd}
TTM \arrow[r, "J_M"] \arrow[d, "p_{TM}"'] & TTM \arrow[d, "Tp_M"] \\
TM \arrow[r, "\mathrm{Id}_{TM}"'] & TM
\end{tikzcd}
\tag{2.2}
$$

which is given as follows [21, 32]. Elements $(\xi; v, x; m) \in TTM$, that is, with $p_{TM}(\xi) = v \in T_mM$ and $Tp_M(\xi) = x \in T_mM$, are considered as second derivatives

$$\xi = \frac{\partial^2 \sigma}{\partial t \partial u}(0,0),$$

where $\sigma : \mathbb{R}^2 \to M$ is a smooth square of elements of M. The notation means that σ is first differentiated with respect to u, yielding a curve $v(t) = \frac{\partial \sigma}{\partial u}(t,0)$ in TM with $\frac{d}{dt}\big|_{t=0} v(t) = \xi$. Thus, $v = \frac{\partial \sigma}{\partial u}(0,0) = p_{TM}(\xi)$ and $x = \frac{\partial \sigma}{\partial t}(0,0) = Tp_M(\xi)$. The canonical involution $J_M : TTM \to TTM$ is defined by

$$J_M(\xi) := \frac{\partial^2 \sigma}{\partial u \partial t}(0,0).$$

Now we can apply the tangent functor to the anchor map $\rho : A \longrightarrow TM$, and then compose with the canonical involution to obtain a bundle map $\rho_{TA} : TA \longrightarrow TTM$ defined by

$$\rho_{TA} = J_M \circ T\rho.$$

This defines the tangent anchor map. In order to define the tangent Lie bracket, we observe that every section $a \in \Gamma_M(A)$ induces two types of sections of $TA \longrightarrow TM$. The first type of section is $Ta : TM \longrightarrow TA$, called **linear** section[3], which is given by applying the tangent functor to the section $a : M \longrightarrow A$. The second type of section is the **core** section $\hat{a} : TM \longrightarrow TA$, which is defined by

$$\hat{a}(v_m) = T_m 0^A(v_m) +_{p_A} \overline{a(m)}, \tag{2.3}$$

where $0^A : M \longrightarrow A$ denotes the zero section, and $\overline{a(m)} = \frac{d}{dt}\big|_{t=0} ta(m) \in T_{0^A_m}A$. As observed in [22], sections of the form Ta and $\hat{a}$ generate the module of sections $\Gamma_{TM}(TA)$. Therefore, the tangent Lie bracket $[\cdot\,,\cdot]_{TA}$ is completely determined by

$$[Ta, Tb]_{TA} = T[a,b], \quad \left[Ta, \hat{b}\right]_{TA} = \widehat{[a,b]}, \quad \left[\hat{a}, \hat{b}\right]_{TA} = 0$$

for all $a, b \in \Gamma(A)$, the extension to general sections is done using the Leibniz rule with respect to the tangent anchor ρ_{TA}.

[3] Note that not all linear sections are of this type: if $\phi : TM \to A$ is a vector bundle homomorphism over Id_M, and $a \in \Gamma(A)$, then the section a^ϕ of $TA \to TM$ defined by

$$a^\phi(v_m) = Ta(v_m) + \frac{d}{dt}\bigg|_{t=0} a_m + t \cdot \phi(v_m)$$

for all $v_m \in TM$ is also a linear section, i.e., $a^\phi : TM \to TA$ is a vector bundle homomorphism over $a : M \to A$.

2.2.2. *The Lie algebroid of the tangent groupoid.* If A is now the Lie algebroid of a Lie groupoid $G \rightrightarrows M$, then we can consider the Lie algebroid $q_{A(TG)} : A(TG) \to TM$ of the tangent Lie groupoid $TG \rightrightarrows TM$. Since the projection $p_G : TG \to G$ is a Lie groupoid morphism, we have a Lie algebroid morphism $A(p_G) : A(TG) \to A$ over $p_M : TM \to M$ and the following diagram commutes.

$$\begin{tikzcd} A(TG) \arrow[r, "A(p_G)"] \arrow[d, "q_{A(TG)}"'] & A \arrow[d, "q_A"] \\ TM \arrow[r, "p_M"'] & M \end{tikzcd} \tag{2.4}$$

Let a be a section of the Lie algebroid A, choose $v \in TM$ and consider the curve $\gamma : (-\varepsilon, \varepsilon) \to TG$ defined by

$$\gamma(t) = T\,\mathrm{Exp}(ta)v$$

for ε small enough. Then we have $\gamma(0) = v$ and $T\mathsf{s}(\gamma(t)) = v$ for all $t \in (-\varepsilon, \varepsilon)$. Hence, $\dot\gamma(0) \in A_v(TG)$ and we can define a **linear** section $\beta_a : TM \to A(TG)$ by

$$\beta_a(v) = \left.\frac{d}{dt}\right|_{t=0} T\,\mathrm{Exp}(ta)v \tag{2.5}$$

for all $v \in TM$. It is easy to check that $\beta_a^r \in \mathfrak{X}(TG)^r$ is the complete lift of a^r (see [21]). In particular the flow of β_a^r is $TL_{Exp(\cdot a)}$, and (β_a, a) is a morphism of vector bundles.

In the same manner, we can consider $v \in TM$, $a \in A_{p_M(v)}$ and the curve $\gamma : \mathbb{R} \to TG$ defined by

$$\gamma(t) = v + ta,$$

where TM and A are seen as subsets of TG, $T_M G = TM \oplus A$. We have again $\gamma(0) = v$ and $T\mathsf{s}(\gamma(t)) = v$ for all t, which yields $\dot\gamma(0) \in A_v(TG)$. Given $a \in \Gamma_M(A)$, we define a **core** section $\tilde a$ of $A(TG)$ by

$$\tilde a(v) = \left.\frac{d}{dt}\right|_{t=0} v + ta(p_M(v)) \tag{2.6}$$

for all $v \in TM$. We have for $v_g \in T_gG$ with $T_g\mathsf{t}(v_g) = v_m$:

$$\tilde a^r(v_g) = \tilde a(v_m) \star 0_{v_g} = \left.\frac{d}{dt}\right|_{t=0} v_g + ta^r(g).$$

The vector bundle $A(TG)$ is spanned by the two types of sections β_a and $\tilde a$, for $a \in \Gamma_M(A)$, and, using the flows of β_a^r and $\tilde b^r \in \mathfrak{X}^r(TG)$, it is easy to check that the equalities

$$[\beta_a, \beta_b]_{A(TG)} = \beta_{[a,b]}, \quad \left[\beta_a, \tilde b\right]_{A(TG)} = \widetilde{[a,b]}, \quad \left[\tilde a, \tilde b\right]_{A(TG)} = 0$$

hold for all $a, b \in \Gamma_M(A)$.

There exists a natural injective bundle map

$$\iota_A : A \longrightarrow TG \tag{2.7}$$

over $\epsilon : M \to G$. The canonical involution $J_G : TTG \longrightarrow TTG$ restricts to an isomorphism of Lie algebroids $j_G : TA \longrightarrow A(TG)$. More precisely, there exists a commutative diagram

$$\begin{tikzcd} TA \arrow[r, "j_G"] \arrow[d, "T\iota_A"'] & A(TG) \arrow[d, "\iota_{A(TG)}"] \\ TTG \arrow[r, "J_G"'] & TTG \end{tikzcd} \tag{2.8}$$

We get easily the following identities

$$j_G \circ Ta = \beta_a \qquad \text{and} \qquad j_G \circ \hat{a} = \tilde{a},$$

where $\hat{a}$ is defined as in (2.3). The equality

$$\rho_{A(TG)} \circ j_G = J_M \circ T\rho_A = \rho_{TA}$$

is verified on linear and core sections. This shows that the Lie algebroid $A(TG) \to TM$ of the tangent groupoid is canonically isomorphic to the tangent Lie algebroid $TA \to TM$ of A.

2.2.3. *The standard Lie algebroid $TA \to A$.* We study now the Lie algebroid structure $TA \to A$ in terms of special sections. Consider a linear vector field on A, i.e., a section X of $TA \to A$ such that the map

$$\begin{tikzcd} A \arrow[r, "X"] \arrow[d, "q_A"'] & TA \arrow[d, "Tq_A"] \\ M \arrow[r, "\bar{X}"'] & TM \end{tikzcd}$$

is a vector bundle morphism over $\bar{X} \in \mathfrak{X}(M)$. For any $b \in \Gamma_M(A)$, the core section $b^\uparrow$ of $TA \to A$ associated to b is defined by

$$b^\uparrow(a_m) = \left.\frac{d}{dt}\right|_{t=0} a_m + tb(m) \tag{2.9}$$

for all $a_m \in A$.

The flow ϕ^X of X is then a vector bundle morphism over the flow $\phi^{\bar{X}}$ of $\bar{X}$

$$\begin{tikzcd} A \arrow[r, "\phi_t^X"] \arrow[d, "q_A"'] & A \arrow[d, "q_A"] \\ M \arrow[r, "\phi_t^{\bar{X}}"'] & M \end{tikzcd}$$

for $t \in \mathbb{R}$ such that this makes sense. The flow of $b^\uparrow$ is given by $\phi_t^{b^\uparrow}(a_m) = a_m + tb(m)$ for all $a_m \in A$ and all $t \in \mathbb{R}$. In particular, for every $t \in \mathbb{R}$, $\phi_t^{b^\uparrow}$ is a vector bundle automorphism of A, covering the identity.

$$\begin{tikzcd} A \arrow[r, "\phi_t^{b^\uparrow}"] \arrow[d, "q_A"'] & A \arrow[d, "q_A"] \\ M \arrow[r, "\mathrm{Id}_M"'] & M \end{tikzcd}$$

Lemma 2.1. *Let A be a Lie algebroid. Choose linear sections $(X, \bar{X}), (Y, \bar{Y})$ of $TA \to A$ and sections $a, b \in \Gamma_M(A)$. Then*

$$\left([X, Y], \left[\bar{X}, \bar{Y}\right]\right)$$

is a linear section of $TA \to A$,

$$\left[a^\uparrow, b^\uparrow\right] = 0$$

and

$$\left[X, a^\uparrow\right] = (D_X a)^\uparrow,$$

where

$$D_X a \in \Gamma(A), \quad (D_X a)(m) = \left.\frac{d}{dt}\right|_{t=0} \underbrace{\left(\phi^X_{-t} \circ a \circ \phi^{\bar{X}}_t\right)(m)}_{\in A_m}$$

for all $m \in M$.

Proof. This is easy to show using the flows of linear and core sections, see also [19]. □

Recall that if $G \rightrightarrows M$ is a Lie groupoid, then the tangent space TG is spanned outside of M by star sections $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$, i.e. with $X \in \mathfrak{X}(G)$, $\bar{X} \in \mathfrak{X}(M)$ such that $X \sim_{\mathsf{s}} \bar{X}$ and $\bar{X} \sim_{\epsilon} X$ (see Subsection 2.1.4). The bracket of two star sections is a star section and for any $a_m \in A$, the properties of the star vector field $X \in \mathfrak{X}(G)$ over $\bar{X} \in \mathfrak{X}(M)$ imply that $(TX)(a_m)$ has value in $A_{\bar{X}(m)}(TG)$. Hence, we can mimic the construction of the Lie algebroid map associated to a Lie groupoid morphism and we can consider the map $A(X) : A \to A(TG)$, $A(X)(a_m) = T_m X(a_m)$ over $\bar{X} : M \to TM$. In the same manner, if $a \in \Gamma_M(A)$, then we can define $\bar{a} : A \to A(TG)$ by $\bar{a}(b_m) = \beta_b(0^{TM}_m) +_{q_{A(TG)}} \tilde{a}(0^{TM}_m)$ for any section $b \in \Gamma_M(A)$ such that $b(m) = b_m$. We have $j_G^{-1} \circ \bar{a} = a^{\uparrow} : A \to TA$ and for a star section $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$ of $TG \to TM$, the map $\tilde{X} := j_G^{-1} \circ A(X) : A \to TA$ is a linear section $(\tilde{X}, \bar{X})$ of $TA \to A$. There is a unique Lie algebroid structure on $A(TG)$ over A making $j_G : TA \to A(TG)$ into a Lie algebroid isomorphism over the identity on A [19, 22, 20].

2.3. **Flat partial connections.** The following definition will be crucial in this paper.

Definition 2.2. *([2]) Let M be a smooth manifold and $F \subseteq TM$ a smooth involutive vector subbundle of the tangent bundle. Let $E \to M$ be a vector bundle over M. A F-**partial connection** is a map $\nabla : \Gamma(F) \times \Gamma(E) \to \Gamma(E)$, written $\nabla(X, e) =: \nabla_X e$ for $X \in \Gamma(F)$ and $e \in \Gamma(E)$, such that:*

1. *∇ is tensorial in the F-argument,*
2. *∇ is $\mathbb{R}$-linear in the E-argument,*
3. *∇ satisfies the Leibniz rule*

$$\nabla_X(fe) = X(f)e + f\nabla_X e$$

for all $X \in \Gamma(F)$, $e \in \Gamma(E)$, $f \in C^\infty(M)$.

*The connection is **flat** if*

$$\nabla_{[X,Y]} e = (\nabla_X \nabla_Y - \nabla_Y \nabla_X)\, e$$

for all $X, Y \in \Gamma(F)$ and $e \in \Gamma(E)$.

Example 2.3 (The Bott connection)**.** Let M be a smooth manifold and $F \subseteq TM$ an involutive subbundle. The **Bott connection**

$$\nabla^F : \Gamma(F) \times \Gamma(TM/F) \to \Gamma(TM/F)$$

defined by

$$\nabla_X \bar{Y} = \overline{[X, Y]},$$

where $\bar{Y} \in \Gamma(TM/F)$ is the projection of $Y \in \mathfrak{X}(M)$, is a flat F-partial connection on $TM/F \to M$.

The class $\bar{Y} \in \Gamma(TM/F)$ of a vector field is ∇^F-parallel if and only if $[Y, \Gamma(F)] \subseteq \Gamma(F)$. Since F is involutive, this does not depend on the representative of $\bar{Y}$. We say by abuse of notation that Y is ∇^F-parallel.

The following proposition can be easily shown by using the fact that the parallel transport defined by a flat connection does not depend on the chosen path in simply connected sets (see [15], [16] for similar statements).

Proposition 2.4. *Let $E \to M$ be a smooth vector bundle of rank k, $F \subseteq TM$ an involutive subbundle and ∇ a flat partial F-connection on E. Then there exists for each point $m \in M$ a frame of local ∇-parallel sections $e_1, \ldots, e_k \in \Gamma(E)$ defined on an open neighborhood U of m in M.*

We have also the following lemma. The first statement is a straightforward consequence of the Leibniz identity and the proof of the second statement can be checked easily in coordinates adapted to the foliation.

Lemma 2.5. *Let $E \to M$ be a smooth vector bundle of rank k, $F \subseteq TM$ an involutive subbundle and ∇ a partial F-connection on E.*

(1) *Assume that $f \in C^\infty(M)$ is F-invariant, i.e., $X(f) = 0$ for all $X \in \Gamma(F)$. Then $f \cdot e$ is ∇-parallel for any ∇-parallel section $e \in \Gamma(E)$.*
(2) *Assume that the foliation defined by F on M is simple, i.e. the leaf space has a smooth manifold structure such that the quotient $\pi : M \to M/F$ is a smooth surjective submersion. Then $X \in \mathfrak{X}(M)$ is ∇^F-parallel if and only if there exists $\bar{X} \in \mathfrak{X}(M/F)$ such that $X \sim_\pi \bar{X}$.*

3. Foliated groupoids

3.1. **Definition and properties.**

Definition 3.1. *Let $G \rightrightarrows M$ be a Lie groupoid. A subbundle $F_G \subseteq TG$ is **multiplicative** if it is a subgroupoid of $TG \rightrightarrows TM$ over $F_G \cap TM =: F_M$. We say that F_G is a **multiplicative foliation** on G if it is involutive and multiplicative. The pair $(G \rightrightarrows M, F_G)$ is then called a **foliated groupoid**.*

Remark 3.2. Multiplicative subbundles were introduced in [31] as follows. A subbundle $F_G \subseteq TG$ is multiplicative if for all composable $g, h \in G$ and $u \in F_G(g \star h)$, there exist $v \in F_G(g)$, $w \in F_G(h)$ such that $u = v \star w$. It is easy to check that a multiplicative foliation in the sense of Definition 3.1 is multiplicative in the sense of [31], but the converse is not necessarily true, unless for instance if the Lie groupoid is a Lie group (see [14]). The case of involutive wide subgroupoids of $TG \rightrightarrows TM$ has also been studied in [1].

The following result about multiplicative subbundles can be found in [15].

Lemma 3.3. *Let $G \rightrightarrows M$ be a Lie groupoid and $F_G \subseteq TG$ a multiplicative subbundle. Then the intersection $F_M := F_G \cap TM$ has constant rank on M. Since it is the set of units of F_G seen as a subgroupoid of TG, the pair $F_G \rightrightarrows F_M$ is a Lie groupoid.*

The bundle $F_G|_M$ splits as $F_G|_M = F_M \oplus F_{\mathsf{c}}$, where $F_{\mathsf{c}} := F_G \cap A$. We have

$$(F_G \cap T^{\mathsf{s}}G)(g) = F_{\mathsf{c}}(\mathsf{t}(g)) \star 0_g = T_{\mathsf{t}(g)} r_g \left(F_{\mathsf{c}}(\mathsf{t}(g))\right)$$

for all $g \in G$.

In the same manner, if $F^{\mathsf{t}} := (F_G \cap T^{\mathsf{t}}G)|_M$, we have $(F_G \cap T^{\mathsf{t}}G)(g) = 0_g \star F^{\mathsf{t}}(\mathsf{s}(g))$ for all $g \in G$.

As a consequence, the intersections $F_G \cap T^{\mathsf{t}}G$ and $F_G \cap T^{\mathsf{s}}G$ have constant rank on G.

The previous lemma says that a multiplicative subbundle $F_G \subseteq TG$ determines a $\mathcal{VB}$-groupoid

$$\begin{tikzcd}
F_G \arrow[r, "p_G"] \arrow[d, shift right, "T\mathsf{t}"'] \arrow[d, shift left, "T\mathsf{s}"] & G \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] \\
F_M \arrow[r, "p_M"'] & M
\end{tikzcd}$$

with **core** F_{c}. As a corollary, one gets that the source and target maps of $F_G \rightrightarrows F_M$ are fiberwise surjective.

Corollary 3.4. *Let $G \rightrightarrows M$ be a Lie groupoid and F_G a multiplicative subbundle of TG. The induced maps $T_g\mathsf{s} : F_G(g) \to F_G(\mathsf{s}(g)) \cap T_{\mathsf{s}(g)}M$ and $T_g\mathsf{t} : F_G(g) \to F_G(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}M$ are surjective for each $g \in G$.*

Proposition 3.5. *Let F_G be a multiplicative foliation on a Lie groupoid $G \rightrightarrows M$. Then F_{c} is a subalgebroid of A and F_M is an involutive subbundle of TM.*

Proof. Choose first two sections $a, b \in \Gamma(F_{\mathsf{c}})$. Then we have $a^r, b^r \in \Gamma(F_G)^r$ by Lemma 3.3 and hence $[a, b]^r = [a^r, b^r] \in \Gamma(F_G)$ since F_G is involutive. Again by Lemma 3.3, we find thus that $[a, b] \in \Gamma(F_{\mathsf{c}})$.

Choose now two sections $\bar{X}, \bar{Y} \in \Gamma(F_M)$ and $X, Y \in \Gamma(F_G)$ that are s-related to $\bar{X}, \bar{Y}$. This is possible by Corollary 3.4. We find then that $[X, Y] \in \Gamma(F_G)$ and $[X, Y] \sim_{\mathsf{s}} [\bar{X}, \bar{Y}]$. Hence, $[\bar{X}, \bar{Y}] \in \Gamma(F_M)$. □

As explained in [12, 26], a multiplicative foliation is a $\mathcal{L}A$-groupoid in the sense of [20],

$$\begin{tikzcd}
F_G \arrow[rr, shift left, "T\mathsf{s}"] \arrow[rr, shift right, "T\mathsf{t}"'] \arrow[dr, "\iota_{F_G}"] \arrow[dd, "p_G"'] & & F_M \arrow[dr, "\iota_{F_M}"] \arrow[dd, dashed] & \\
& TG \arrow[rr, shift left, "T\mathsf{s}"] \arrow[rr, shift right, "T\mathsf{t}"'] \arrow[dl, "p_G"'] & & TM \arrow[dl, "p_M"] \\
G \arrow[rr, shift left, "\mathsf{s}"] \arrow[rr, shift right, "\mathsf{t}"'] & & M &
\end{tikzcd}$$

where and ι_{F_M}, ι_{F_G} are the inclusions of F_M in TM and of F_G in TG, respectively. That is, $(F_M \to M, \iota_{F_M}, [\cdot\,,\cdot])$ is a Lie algebroid over M and the quadruple $(F_G; G, F_M; M)$ is such that F_G has both a Lie groupoid structure over F_M and a Lie algebroid structure over G such that the two structures on F_G are compatible in the sense that the maps defining the groupoid structure are all Lie algebroid morphisms. Furthermore, the double source map

$$(p_G, T\mathsf{s}) : F_G \to G \times_M F_M = \{(g, v) \in G \times F_M \mid v \in F_M(\mathsf{s}(g))\}$$

is a surjective submersion by Corollary 3.4.

3.2. **The connection associated to a foliated groupoid.** Recall that if $G \rightrightarrows M$ is a Lie groupoid and F_G is a multiplicative foliation on G, then $F_M := F_G \cap TM$ and $F_{\mathsf{c}} = F_G \cap A$ are subalgebroids of $TM \to M$ and $A \to M$, respectively (Proposition 3.5).

In the main theorem of this subsection, we show that if F_G is a multiplicative foliation on the Lie groupoid $G \rightrightarrows M$, then the Bott F_G-connection on TG/F_G induces a well-defined partial F_M-connection on A/F_{c}. We write $\bar{a}$ for the class in A/F_{c} of $a \in \Gamma(A)$.

Theorem 3.6. *Let $(G \rightrightarrows M, F_G)$ be a Lie groupoid endowed with a multiplicative foliation. Then there is a partial F_M-connection on A/F_{c}*

$$\nabla : \Gamma(F_M) \times \Gamma\left(A/F_{\mathsf{c}}\right) \to \Gamma\left(A/F_{\mathsf{c}}\right). \tag{3.10}$$

with the following properties:

1. *∇ is flat.*
2. *If $a \in \Gamma(A)$ is ∇-parallel, i.e., $\nabla_{\bar{X}}\bar{a} = 0$ for all $\bar{X} \in \Gamma(F_M)$, then $[a, b] \in \Gamma(F_{\mathsf{c}})$ for all $b \in \Gamma(F_{\mathsf{c}})$.*
3. *If $a, b \in \Gamma(A)$ are ∇-parallel, then $[a, b]$ is also ∇-parallel.*
4. *If $a \in \Gamma(A)$ is ∇-parallel, then $[\rho(a), \bar{X}] \in \Gamma(F_M)$ for all $\bar{X} \in \Gamma(F_M)$. That is, $\rho(a) \in \mathfrak{X}(M)$ is ∇^{F_M}-parallel.*

Furthermore, $a \in \Gamma(A)$ is ∇-parallel if and only if the right invariant vector field $a^r \in \mathfrak{X}(G)$ is parallel with respect to the Bott connection defined by F_G, that is

$$[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G).$$

In the following, we call a vector field $X \in \mathfrak{X}(G)$ a **t-section** if there exists $\bar{X} \in \mathfrak{X}(M)$ such that $X \sim_{\mathsf{t}} \bar{X}$. Similarly, a one-form $\eta \in \Omega^1(G)$ is a t-section of T^*G if $\tilde{\mathsf{t}} \circ \eta = \bar{\eta} \circ \mathsf{t}$ for some $\bar{\eta} \in \Gamma(A^*)$. Analogously, we can define **s-sections** of TG and T^*G.

It is easy to see that F_G and its annihilator ${F_G}^\circ \subseteq T^*G$ are spanned by their t-sections.

Lemma 3.7. *Let $(G \rightrightarrows M, F_G)$ be a Lie groupoid endowed with a multiplicative subbundle $F_G \subseteq TG$. Choose $\bar{X} \in \Gamma(F_M)$, $\bar{\eta} \in \Gamma(F_G^\circ \cap A^*)$ and t-sections $X \sim_{\mathsf{t}} \bar{X}$, $\eta \sim_{\tilde{\mathsf{t}}} \bar{\eta}$ of F_G and F_G°, respectively. Then the identity*

$$\eta(\pounds_{a^r} X) = \mathsf{t}^*\epsilon^*\big(\bar{\eta}(\pounds_{a^r} X)\big) \tag{3.11}$$

holds for any section $a \in \Gamma(A)$.

Proof. Choose $g \in G$ and set $p = \mathsf{t}(g)$. For all $t \in (-\varepsilon, \varepsilon)$ for a small ε, we have

$$X(\mathrm{Exp}(ta)(p) \star g) = (X(\mathrm{Exp}(ta)(p))) \star \big(X(\mathrm{Exp}(ta)(p))\big)^{-1} \star X(\mathrm{Exp}(ta)(p) \star g).$$

The vector

$$\big(X(\mathrm{Exp}(ta)(p))\big)^{-1} \star X(\mathrm{Exp}(ta)(p) \star g)$$

is an element of $F_G(g)$ for all $t \in (-\varepsilon, \varepsilon)$ and will be written $v_t(g)$ to simplify the notation. We compute

$$\begin{aligned}
\eta(\pounds_{a^r} X)(g) &= \eta(g)\left(\left.\frac{d}{dt}\right|_{t=0} \big(T_{L_{\mathrm{Exp}(ta)}(g)} L_{\mathrm{Exp}(-ta)} X(\mathrm{Exp}(ta)(p) \star g)\big)\right)\\
&= \left.\frac{d}{dt}\right|_{t=0} (\bar{\eta}(p) \star \eta(g))\Big(\big(T_{\mathrm{Exp}(ta)} L_{\mathrm{Exp}(-ta)} X(\mathrm{Exp}(ta)(p))\big) \star v_t(g)\Big)\\
&= \left.\frac{d}{dt}\right|_{t=0} \bar{\eta}(p)\big(T_{\mathrm{Exp}(ta)} L_{\mathrm{Exp}(-ta)} X(\mathrm{Exp}(ta)(p))\big) + \left.\frac{d}{dt}\right|_{t=0} \eta(g)(v_t(g))\\
&= \bar{\eta}(\pounds_{a^r} X)(p) + \left(\left.\frac{d}{dt}\right|_{t=0} 0\right) = \bar{\eta}(\pounds_{a^r} X)(\mathsf{t}(g)).
\end{aligned}$$

□

Let $G \rightrightarrows M$ be a Lie groupoid and F_G a multiplicative subbundle of TG. Recall that the restriction to M of F_G splits as

$$F_G|_M = F_M \oplus F_{\mathsf{c}}, \tag{3.12}$$

where $F_{\mathsf{c}} = F_G \cap A$ and $F_M = F_G \cap TM$. Recall also the notation $F^{\mathsf{t}} = F_G \cap T_M^{\mathsf{t}} G$.

Theorem 3.8. *Let $(G \rightrightarrows M, F_G)$ be a Lie groupoid endowed with a multiplicative subbundle $F_G \subseteq TG$, X a t-section of F_G, i.e., t-related to some $\bar{X} \in \Gamma(F_M)$, and consider $a \in \Gamma(A)$. Then the derivative $\pounds_{a^r} X$ can be written as a sum*

$$\pounds_{a^r} X = Z_{a,X} + b_{a,X}^r$$

with $b_{a,X} \in \Gamma(A)$, and $Z_{a,X}$ a t-section of F_G. In addition, if $X \sim_{\mathsf{t}} 0$, then $\pounds_{a^r} X \in \Gamma(F_G \cap T^{\mathsf{t}} G)$. In particular, its restriction to M is a section of F^{t} and $b_{a,X}$ is a section of F_{c}.

Proof. Set

$$b_{a,X}(m) = \pounds_{a^r} X(m) - T_m \mathsf{s}\left(\pounds_{a^r} X(m)\right)$$

for all $m \in M$ and

$$Z_{a,X} := \pounds_{a^r} X - b_{a,X}^r.$$

In particular, $Z_{a,X}(m) = T_m\mathsf{s}\left(\pounds_{a^r}X(m)\right)$ for all $m \in M$. First, we see that $Z_{a,X}$ is a t-section of TG since

$$Z_{a,X} = \pounds_{a^r}X - b^r_{a,X} \sim_{\mathsf{t}} \left[\rho(a), \bar{X}\right] - \rho(b_{a,X}).$$

Recall that $F_G{}^\circ \subseteq T^*G$ is spanned by its t-sections. Hence, to show that $Z_{a,X} \in \Gamma(F_G)$, it suffices to show that $\eta(Z_{a,X}) = 0$ for all t-sections η of $F_G{}^\circ$.

Setting $\mathsf{t}(g) = p$, we have immediately

$$\begin{aligned} \eta(Z_{a,X})(g) &= \eta(\pounds_{a^r}X)(g) - \bar{\eta}(b_{a,X})(p) \\ &\overset{(3.11)}{=} \bar{\eta}(\pounds_{a^r}X)(p) - \bar{\eta}(b_{a,X})(p) \\ &= \bar{\eta}(T_p\mathsf{s}(\pounds_{a^r}X(p))) = 0 \end{aligned}$$

since $\bar{\eta}$ vanishes on TM. This shows that $Z_{a,X}$ is a section of F_G.

Assume now that $X \sim_{\mathsf{t}} 0$. This means that $X \in \Gamma(F_G \cap T^{\mathsf{t}}G)$ and X can be written $X = \sum_{i=1}^r f_i \cdot b_i^l$ with $f_1, \ldots, f_r \in C^\infty(G)$ and $b_1, \ldots, b_r \in \Gamma(F^{\mathsf{t}})$. We get then

$$\pounds_{a^r}X = \pounds_{a^r}\left(\sum_{i=1}^r f_i \cdot b_i^l\right) = \sum_{i=1}^r a^r(f_i)b_i^l + \sum_{i=1}^r f_i \cdot \left[a^r, b_i^l\right] = \sum_{i=1}^r a^r(f_i)b_i^l,$$

which is again a section of $F_G \cap T^{\mathsf{t}}G$. □

Assume now that F_G is involutive and define

$$\nabla : \Gamma(F_M) \times \Gamma(A/F_{\mathsf{c}}) \to \Gamma(A/F_{\mathsf{c}})$$

by

$$\nabla_{\bar{X}}\bar{a} = -\overline{b_{a,X}},$$

with $b_{a,X}$ as in Theorem 3.8, for any choice of t-section $X \in \Gamma(F_G)$ such that $X \sim_{\mathsf{t}} \bar{X}$ and any choice of representative $a \in \Gamma(A)$ for $\bar{a}$. We will show that this is a well-defined partial F_M-connection and complete the proof of Theorem 3.6.

Proof of Theorem 3.6. Choose $X, X' \in \Gamma(F_G)$ such that $X \sim_{\mathsf{t}} \bar{X}$ and $X' \sim_{\mathsf{t}} \bar{X}$. Then $Y := X - X' \sim_{\mathsf{t}} 0$ and, by Theorem 3.8, we find $b_{a,Y} \in \Gamma(F_{\mathsf{c}})$ for any $a \in \Gamma(A)$, i.e., $\overline{b_{a,X}} = \overline{b_{a,X'}}$.

Choose now $a \in \Gamma(F_{\mathsf{c}})$ and $X \in \Gamma(F_G)$, $X \sim_{\mathsf{t}} \bar{X} \in \Gamma(F_M)$. Then we have $a^r \in \Gamma(F_G)$ and since F_G is involutive, $\pounds_{a^r}X \in \Gamma(F_G)$. Again, since $Z_{a,X} \in \Gamma(F_G)$, we find $b_{a,X} \in \Gamma(F_{\mathsf{c}})$. This shows that ∇ is well-defined.

By definition, if $a \in \Gamma(A)$ is such that $\nabla_{\bar{X}}\bar{a} = 0$ for all $\bar{X} \in \Gamma(F_M)$, then we have $\pounds_{a^r}X = Z_{a,X} + b^r_{a,X} \in \Gamma(F_G)$ for all t-descending sections $X \in \Gamma(F_G)$. Since $\Gamma(F_G)$ is spanned as a $C^\infty(G)$-module by its t-descending sections, we get

$$[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G).$$

Conversely, $[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G)$ implies immediately $\nabla_{\bar{X}}\bar{a} = 0$ for all $\bar{X} \in \Gamma(F_M)$. This proves the second claim of the theorem.

We check that ∇ is a flat partial F_M-connection. Choose $a \in \Gamma(A)$, $\bar{X} \in \Gamma(F_M)$, $X \in \Gamma(F_G)$ such that $X \sim_{\mathsf{t}} \bar{X}$ and $f \in C^\infty(M)$. Then we have $\mathsf{t}^*f \cdot X \sim_{\mathsf{t}} f\bar{X}$ and

$$\pounds_{a^r}(\mathsf{t}^*f \cdot X) = \mathsf{t}^*(\rho(a)(f)) \cdot X + \mathsf{t}^*f \cdot \pounds_{a^r}X.$$

In particular, we find

$$\begin{aligned} b_{a,\mathsf{t}^*f\cdot X} &= (1 - T\mathsf{s})\left(\mathsf{t}^*(\rho(a)(f)) \cdot X + \mathsf{t}^*f \cdot \pounds_{a^r}X\right)|_M \\ &= \rho(a)(f) \cdot (1 - T\mathsf{s})X|_M + f \cdot (1 - T\mathsf{s})(\pounds_{a^r}X)|_M. \end{aligned}$$

Since $(T\mathsf{s}-1)X|_M\in\Gamma(F_{\mathsf{c}})$, this leads to $\overline{b_{a,\mathsf{t}^*f\cdot X}}=f\cdot\overline{b_{a,X}}$ and hence $\nabla_{f\bar X}\bar a=-f\cdot\overline{b_{a,X}}=f\cdot\nabla_{\bar X}\bar a$.

Since $(fa)^r=\mathsf{t}^*f\cdot a^r$, we have in the same manner

$$\pounds_{(fa)^r}X=-\pounds_X(\mathsf{t}^*f\cdot a^r)=-\mathsf{t}^*(\bar X(f))\cdot a^r+\mathsf{t}^*f\cdot\pounds_{a^r}X,$$

which leads to $\nabla_{\bar X}(f\cdot\bar a)=\bar X(f)\cdot\bar a+f\cdot\nabla_{\bar X}\bar a$.

Choose $\bar X,\bar Y\in\Gamma(F_M)$ and $X,Y\in\Gamma(F_G)$ such that $X\sim_{\mathsf{t}}\bar X$ and $Y\sim_{\mathsf{t}}\bar Y$. Then we have $[X,Y]\sim_{\mathsf{t}}[\bar X,\bar Y]$ and $[X,Y]\in\Gamma(F_G)$ since F_G is involutive. For any $a\in\Gamma(A)$, we have by the Jacobi-identity:

$$\begin{aligned}\pounds_{a^r}[X,Y]&=[\pounds_{a^r}X,Y]-[\pounds_{a^r}Y,X]\\&=\left[Z_{a,X}+b^r_{a,X},Y\right]-\left[Z_{a,Y}+b^r_{a,Y},X\right]\\&=[Z_{a,X},Y]-[Z_{a,Y},X]+\pounds_{b^r_{a,X}}Y-\pounds_{b^r_{a,Y}}X\\&=[Z_{a,X},Y]-[Z_{a,Y},X]+Z_{b_{a,X},Y}+{b_{b_{a,X},Y}}^r-Z_{b_{a,Y},X}-{b_{b_{a,Y},X}}^r.\end{aligned}$$

Since $[Z_{a,X},Y]-[Z_{a,Y},X]+Z_{b_{a,X},Y}-Z_{b_{a,Y},X}$ is a t-section of F_G, we find that

$$\nabla_{[\bar X,\bar Y]}\bar a=\overline{b_{b_{a,Y},X}}-\overline{b_{b_{a,X},Y}}=\nabla_{\bar X}\nabla_{\bar Y}\bar a-\nabla_{\bar Y}\nabla_{\bar X}\bar a$$

which shows the flatness of ∇.

Choose now $a\in\Gamma(A)$ such that $\nabla_{\bar X}\bar a=0\in\Gamma(A/F_{\mathsf{c}})$ for all $\bar X\in\Gamma(F_M)$. If $b\in\Gamma(F_{\mathsf{c}})$, then $b^r\in\Gamma(F_G)$, $\rho(b)\in\Gamma(F_M)$ and $b^r\sim_{\mathsf{t}}\rho(b)$. This leads to

$$\overline{[b,a]}=\nabla_{\rho(b)}\bar a=0\in\Gamma(A/F_{\mathsf{c}})$$

and hence $[a,b]\in\Gamma(F_{\mathsf{c}})$. This shows 2. For each $\bar X\in\Gamma(F_M)$, there exists $X\in\Gamma(F_G)$ such that $X\sim_{\mathsf{t}}\bar X$. Since $[a^r,X]\in\Gamma(F_G)$, $a^r\sim_{\mathsf{t}}\rho(a)$ and $T\mathsf{t}(F_G)=F_M$, we find $[\rho(a),\bar X]\in\Gamma(F_M)$, which proves 4.

To show 3., choose two sections $a,b\in\Gamma(A)$ such that $\bar a$ and $\bar b$ are ∇-parallel. We have then for any t-section $X\sim_{\mathsf{t}}\bar X$ of F_G:

$$\begin{aligned}\pounds_{[a,b]^r}X&=\pounds_{a^r}(Z_{b,X}+b^r_{b,X})-\pounds_{b^r}(Z_{a,X}+b^r_{a,X})\\&=\pounds_{a^r}(Z_{b,X})+[a,b_{b,X}]^r-\pounds_{b^r}(Z_{a,X})-[b,b_{a,X}]^r.\end{aligned}$$

Since $\bar a$ and $\bar b$ are ∇-parallel, this yields $\nabla_{\bar X}\overline{[a,b]}=-\overline{[a,b_{b,X}]}+\overline{[b,b_{a,X}]}$. Since $\bar a$ and $\bar b$ are parallel, we have $b_{b,X},b_{a,X}\in\Gamma(F_{\mathsf{c}})$ and 3. follows using 2. □

Remark 3.9. Let $(G\rightrightarrows M,F_G)$ be a foliated Lie groupoid and consider the multiplicative Dirac structure $F_G\oplus F_G{}^\circ$ on G. The "tangent" part of the Courant algebroid $\mathfrak{B}(F_G\oplus F_G{}^\circ)$ in [12] is in this case just $(F_M\oplus A)/F^{\mathsf{t}}$, which is isomorphic as a vector bundle to $F_M\oplus(A/F_{\mathsf{c}})$. The induced bracket on sections of $F_M\oplus(A/F_{\mathsf{c}})$ defines a map $\Gamma(F_M)\times\Gamma(A/F_{\mathsf{c}})\to\Gamma(A/F_{\mathsf{c}})$,

$$\left(\bar X,\bar a\right)\mapsto\mathrm{pr}_{A/F_{\mathsf{c}}}\left[\left(\bar X,0\right),(0,\bar a)\right],$$

which can be checked to be exactly the connection ∇.

3.3. **Involutivity of a multiplicative subbundle of TG.** It is natural to ask here how the involutivity of F_G is encoded in the data $(F_M,F_{\mathsf{c}},\nabla)$. For an arbitrary (not necessarily involutive) multiplicative subbundle $F_G\subseteq TG$, we can consider the map

$$\tilde\nabla:\Gamma(F_M)\times\Gamma(A)\to\Gamma(A/F_{\mathsf{c}}),$$
$$\tilde\nabla_{\bar X}a=-\overline{b_{a,X}}$$

which is well-defined by the proof of Theorem 3.6.

Theorem 3.10. *Let (G,F_G) be a source-connected Lie groupoid endowed with a multiplicative subbundle. Then F_G is involutive if and only if the following holds:*

(1) *$F_M\subseteq TM$ is involutive,*

(2) $\tilde{\nabla}$ *vanishes on sections of* F_{c},

(3) *the induced map* $\nabla : \Gamma(F_M) \times \Gamma(A/F_{\mathsf{c}}) \to \Gamma(A/F_{\mathsf{c}})$ *is a flat partial* F_M*-connection on* A/F_{c}.

The proof of this theorem is a simplified version of the proof of the general criterion for the closedness of multiplicative Dirac structures (see [13]).

Proof. We have already shown in Proposition 3.5 and Theorem 3.6 that the involutivity of F_G implies (1), (2) and (3).

For the converse implication, recall that the t-star sections of F_G span F_G outside of the set of units M. Hence, it is sufficient to show involutivity on t-star sections and right-invariant sections of F_G. Choose first two right-invariant sections a^r, b^r of F_G, i.e. with $a, b \in \Gamma(F_{\mathsf{c}})$. Then we have $\rho(b) \in \Gamma(F_M)$, $b^r \sim_{\mathsf{t}} \rho(b)$ and, since $\tilde{\nabla}_{\rho(b)} a = 0$ by (2), we find that $[a^r, b^r] \in \Gamma(F_G)$. In the same manner, by the definition of $\tilde{\nabla}$ and Theorem 3.8, Condition (2) implies that the bracket of a right-invariant section of F_G and a t-star section is always a section of F_G.

We have thus only to show that the bracket of two t-star sections of F_G is again a section of F_G. Let F_{c}° be the annihilator of F_{c} in A^* and consider the dual F_M-connection on $(A/F_{\mathsf{c}})^* \simeq {F_{\mathsf{c}}}^{\circ} \subseteq A^*$, i.e. the (by (3)) flat connection

$$\nabla^* : \Gamma(F_M) \times \Gamma({F_{\mathsf{c}}}^{\circ}) \to \Gamma({F_{\mathsf{c}}}^{\circ})$$

given by

$$\left(\nabla^*_{\bar{X}} \alpha\right)(\bar{a}) = \bar{X}(\alpha(\bar{a})) - \alpha\left(\nabla_{\bar{X}} \bar{a}\right)$$

for all $\bar{X} \in \Gamma(F_M)$, $\alpha \in \Gamma({F_{\mathsf{c}}}^{\circ})$ and $a \in \Gamma(A)$.

Choose a t-star section $X \in \Gamma(F_G)$, $X \sim_{\mathsf{t}} \bar{X}$, $X|_M = \bar{X}$ and a $\tilde{\mathsf{t}}$-section $\eta \in \Gamma(F_G^{\circ})$, $\eta \sim_{\tilde{\mathsf{t}}} \bar{\eta}$. Then, for any section a of A, we have

$$\begin{aligned}
(\pounds_X \eta)(a^r) &= X\left(\eta(a^r)\right) + \eta\left(\pounds_{a^r} X\right) \\
&= \mathsf{t}^*\left(\bar{X}(\bar{\eta}(\bar{a}))\right) + \eta\left(Z_{a,X} + b^r_{a,X}\right) \qquad \text{by Theorem 3.8} \\
&= \mathsf{t}^*\left(\bar{X}(\bar{\eta}(\bar{a})) - \bar{\eta}\left(\nabla_{\bar{X}} \bar{a}\right)\right) \qquad \text{since } \eta \in \Gamma(F_G^{\circ}) \text{ and } Z_{a,X} \in \Gamma(F_G) \\
&= \mathsf{t}^*(\nabla^*_{\bar{X}} \bar{\eta}(\bar{a})).
\end{aligned}$$

This shows that $\pounds_X \eta \sim_{\tilde{\mathsf{t}}} \nabla^*_{\bar{X}} \bar{\eta} \in \Gamma(F_{\mathsf{c}}^{\circ})$. Note that we have not shown yet that $\pounds_X \eta$ is a section of F_G°. Choose a second t-star section Y of F_G, $Y \sim_{\mathsf{t}} \bar{Y}$ and $Y|_M = \bar{Y}$. An easy computation using $\eta(X) = \eta(Y) = 0$ yields

$$-2\mathbf{d}\left(\eta([X, Y])\right) = \pounds_X \pounds_Y \eta - \pounds_Y \pounds_X \eta - \pounds_{[X,Y]} \eta.$$

Hence, we get for $a \in \Gamma(A)$:

$$
\begin{aligned}
-2\cdot a^r(\eta([X,Y])) &= \left(\pounds_X\pounds_Y\eta - \pounds_Y\pounds_X\eta - \pounds_{[X,Y]}\eta\right)(a^r)\\
&= X(\pounds_Y\eta(a^r)) - \pounds_Y\eta([X,a^r]) - Y(\pounds_X\eta(a^r)) - \pounds_X\eta([Y,a^r])\\
&\quad - [X,Y](\eta(a^r)) + \eta([[X,Y],a^r])\\
&= \mathsf{t}^*\bar{X}(\nabla^*_{\bar{Y}}\bar{\eta}(\bar{a})) + (\pounds_Y\eta)(Z_{a,X} + b^r_{a,X})\\
&\quad - \mathsf{t}^*\bar{Y}(\nabla^*_{\bar{X}}\bar{\eta}(\bar{a})) - (\pounds_X\eta)(Z_{a,Y} + b^r_{a,Y})\\
&\quad - \mathsf{t}^*[\bar{X},\bar{Y}](\bar{\eta}(\bar{a})) - \eta\left([\pounds_{a^r}X, Y] + [X, \pounds_{a^r}Y]\right)\\
&= \mathsf{t}^*\bar{X}(\nabla^*_{\bar{Y}}\bar{\eta}(\bar{a})) + \pounds_Y\eta(Z_{a,X}) - \mathsf{t}^*(\nabla^*_{\bar{Y}}\bar{\eta})(\nabla_{\bar{X}}\bar{a})\\
&\quad - \mathsf{t}^*\bar{Y}(\nabla^*_{\bar{X}}\bar{\eta}(\bar{a})) - \pounds_X\eta(Z_{a,Y}) + \mathsf{t}^*(\nabla^*_{\bar{X}}\bar{\eta})(\nabla_{\bar{Y}}\bar{a})\\
&\quad - \mathsf{t}^*[\bar{X},\bar{Y}](\bar{\eta}(\bar{a})) - \eta\left([Z_{a,X} + b^r_{a,X}, Y] + [X, Z_{a,Y} + b^r_{a,Y}]\right)\\
&= \mathsf{t}^*\left(\left(\nabla^*_{\bar{X}}\nabla^*_{\bar{Y}}\bar{\eta} - \nabla^*_{\bar{Y}}\nabla^*_{\bar{X}}\bar{\eta}\right)(\bar{a}) - [\bar{X},\bar{Y}](\bar{\eta}(\bar{a}))\right)\\
&\quad + Y(\eta(Z_{a,X})) - X(\eta(Z_{a,Y})) - \eta\left(Z_{b_{a,X},Y} + b^r_{b_{a,X},Y} - Z_{b_{a,Y},X} - b^r_{b_{a,Y},X}\right)\\
&\overset{(3)}{=} \mathsf{t}^*\left(\bar{\eta}(-\nabla_{[\bar{X},\bar{Y}]}\bar{a} - \nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{a} + \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{a})\right) \overset{(3)}{=} 0.
\end{aligned}
$$

Hence, $a^r(\eta([X,Y])) = 0$ for all $a \in \Gamma(A)$ and since G is source-connected, this implies that $\eta([X,Y])(g) = \eta([X,Y])(\mathsf{s}(g))$ for all $g \in G$. But since for $m \in M$, we have

$$[X,Y](m) = [\bar{X},\bar{Y}](m)$$

and F_M is a subalgebroid of TM by (1), we find that $[X,Y](\mathsf{s}(g)) \in F_G(\mathsf{s}(g))$ for all $g \in G$ and hence $\eta([X,Y])(g) = \eta([X,Y])(\mathsf{s}(g)) = 0$. Since η was a $\tilde{\mathsf{t}}$-section of F_G° and $\tilde{\mathsf{t}}$-sections of F_G° span F_G° on G, we have shown that $[X,Y] \in \Gamma(F_G)$ and the proof is complete. □

Remark 3.11. (1) We have seen in this proof that Condition (2) implies the fact that F_{c} is a subalgebroid of A.
(2) The same result has been shown independently in [8], using Lie groupoid and Lie algebroid cocycles, in the special case where $F_M = TM$, i.e. where F_G is a wide subgroupoid of TG.

3.4. **Examples.**

Example 3.12. Assume that G is a Lie group (hence with $M = \{e\}$) with Lie algebra $\mathfrak{g}$. Let F_G be a multiplicative distribution. In this case, the core $F_{\mathsf{c}} =: \mathfrak{f}$ is the fiber of F_G over the identity and $F_M = 0$. As a consequence, any partial F_M-connection on $\mathfrak{g}/\mathfrak{f}$ is trivial. We check that all the conditions in Theorem 3.6 are automatically satisfied.

First of all, any element ξ of $\mathfrak{g}$ is ∇-parallel. This implies that

$$[\xi^r, \Gamma(F_G)] \subseteq \Gamma(F_G) \qquad \text{for all} \qquad \xi \in \mathfrak{g},$$

i.e., F_G is left-invariant, in agreement with [25, 14, 15].

1), 3) and 4) are trivially satisfied and 2) is exactly the fact that $\mathfrak{f}$ is an ideal in $\mathfrak{g}$. This recovers the results proved in [25, 14, 15].

Example 3.13. Let $G \rightrightarrows M$ be a Lie groupoid with a smooth, free and proper action of a Lie group H by Lie groupoid automorphisms. Let $\mathcal{V}_G$ be the vertical space of the action, i.e., the smooth subbundle of TG that is generated by the infinitesimal vector fields ξ_G, for all $\xi \in \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H. The involutive subbundle $\mathcal{V}_G$ is easily seen to be multiplicative (see for instance [15]).

The action restricts to a free and proper action of H on M, and it is easy to check that $\mathcal{V}_G \cap TM = \mathcal{V}_M$ is the vertical vector space of the action of H on M. Furthermore, $\mathcal{V}_G \cap T^{\mathsf{s}}G = \mathcal{V}_G \cap T^{\mathsf{t}}G = 0^{TG}$ and we get $\mathcal{V}^{\mathsf{c}} = 0^A$.

The infinitesimal vector fields (ξ_G, ξ_M), $\xi \in \mathfrak{h}$, are multiplicative (in the sense of [22] for instance). We get hence from [22] that the Lie bracket $[a^r, \xi_G]$ is right-invariant for any $\xi \in \mathfrak{h}$ and $a \in \Gamma(A)$. We obtain a map (see also [22])

$$\begin{array}{ccc} \mathfrak{h} \times \Gamma(A) & \to & \Gamma(A) \\ (\xi, a) & \mapsto & [\xi_G, a^r]|_M \end{array},$$

and we recover the connection

$$\nabla : \quad \Gamma(\mathcal{V}_M) \times \Gamma(A) \quad \to \quad \Gamma(A)$$

defined by $\nabla_{\xi_M} a = [\xi_G, a^r]|_M$ for all $\xi \in \mathfrak{h}$ and $a \in \Gamma(A)$. This connection is obviously flat and satisfies all the conditions in Theorem 3.6.

Example 3.14. Let $(G \rightrightarrows M, J_G)$ be a complex Lie groupoid, i.e., a Lie groupoid endowed with a complex structure J_G that is multiplicative in the sense that the map

$$\begin{tikzcd} TG \arrow[r, "J_G"] \arrow[d, shift right, "T\mathsf{t}"'] \arrow[d, shift left, "T\mathsf{s}"] & TG \arrow[d, shift right, "T\mathsf{t}"'] \arrow[d, shift left, "T\mathsf{s}"] \\ TM \arrow[r, "J_M"'] & TM \end{tikzcd}$$

is a Lie groupoid morphism over some map J_M. Since ${J_G}^2 = -\mathrm{Id}_{TG}$, we conclude that ${J_M}^2 = -\mathrm{Id}_{TM}$ and the Nijenhuis condition for J_M is easy to prove using s-related vector fields. The map J_G restricts also to a map j_A on the core A, i.e., a fiberwise complex structure that satisfies also a Nijenhuis condition. (This can be seen by noting that the Nijenhuis tensor of J_G restricts to right-invariant vector fields.)

The subbundles $T^{1,0}G = E_i$ and $T^{0,1}G = E_{-i}$ of $TG \otimes \mathbb{C}$ are multiplicative and involutive with bases $T^{1,0}M$ and $T^{0,1}M$ and cores $A^{1,0}$ and $A^{0,1}$. The quotient $(A \otimes \mathbb{C})/A^{1,0}$ is isomorphic as a vector bundle to $A^{0,1}$ and a straightforward computation shows that the connection that we get from the multiplicative complex foliation $T^{1,0}G$ is exactly the connection $\nabla : \Gamma(T^{1,0}M) \times \Gamma(A^{0,1}) \to \Gamma(A^{0,1})$ as in Lemma 4.7 of [17].

Since the parallel sections of the connection in this Lemma are exactly the holomorphic sections of $A^{0,1}$, one can reconstruct the map $J_A : TA \to TA$ defined by $J_A = \sigma^{-1} \circ A(J_G) \circ \sigma$ [4] as in [18] by requiring that $J_A(Ta) = Ta \circ J_M$ for all parallel sections $a \in \Gamma(A)$, and $J_A(\hat{b}) = \widehat{j_A(b)} \circ J_M$ for all sections $b \in \Gamma(A)$.

By Lemma 4.7 in [17] and the integration results in [18], the complex structure J_G is hence equivalent to the data J_M, j_A and this connection with its properties. This is in agreement with the results that we will prove in section 5.

4. Foliated algebroids

In this section we study Lie algebroids equipped with foliations compatible with both Lie algebroid structures $TA \to TM$ and $TA \to A$ on TA. This is the first step towards an infinitesimal description of multiplicative foliations.

4.1. **Definition and properties.**

Definition 4.1. *Let $A \to M$ be a Lie algebroid. A subbundle $F_A \subseteq TA$ is called* ***morphic*** *if it is a Lie subalgebroid of $TA \to TM$ over some subbundle $F_M \subseteq TM$.*

If F_A is involutive and morphic, then the pair (A, F_A) is referred to as a ***foliated Lie algebroid****.*

[4] To avoid confusions, we write in this example $\sigma : TA \to A(TG)$ for the canonical flip map.

Consider a foliated Lie algebroid (A, F_A). We have the $\mathcal{VB}$-algebroid

$$\begin{tikzcd} F_A \arrow[r, "Tq_A|_{F_A}"] \arrow[d, "p_A|_{F_A}"'] & F_M \arrow[d, "p_M|_{F_M}"] \\ A \arrow[r, "q_A"'] & M \end{tikzcd}$$

and the Lie algebroid morphisms

$$\begin{tikzcd} F_A \arrow[r, "\iota_{F_A}"] \arrow[d, "Tq_A|_{F_A}"'] & TA \arrow[d, "Tq_A"] \\ F_M \arrow[r, "\iota_{F_M}"'] & TM \end{tikzcd} \qquad \begin{tikzcd} F_A \arrow[r, "\iota_{F_A}"] \arrow[d, "p_A|_{F_A}"'] & TA \arrow[d, "p_A"] \\ A \arrow[r, "\mathrm{Id}_A"'] & A \end{tikzcd} .$$

We have

$$F_A\left(0^A(m)\right) = T_m0^A(F_M(m)) \oplus \ker(Tq_A|_{F_A(0^A(m))}) \tag{4.13}$$

for all $m \in M$. If $v_{0^A(m)} \in F_A(0^A_m)$ is such that $Tq_A(v_{0^A(m)}) = 0_m \in F_M(m)$, then

$$v_{0^A(m)} = \left.\frac{d}{dt}\right|_{t=0} 0^A_m + ta_m$$

for some $a_m \in A_m$. Set

$$F_{\mathsf{c}}(m) := \left\{ a_m \in A_m \,\middle|\, \left.\frac{d}{dt}\right|_{t=0} 0^A_m + ta_m \in F_A\left(0^A_m\right) \right\}$$

for all $m \in M$. Equation (4.13) shows that this defines a vector subbundle of A.

Furthermore, using the fact that F_A is closed under the addition in $TA \to TM$, one can check that

$$\ker\left(Tq_A|_{F_A(b_m)}\right) = \left\{ \left.\frac{d}{dt}\right|_{t=0} b_m + ta_m \,\middle|\, a_m \in F_{\mathsf{c}}(m) \right\} \tag{4.14}$$

for all $b_m \in A_m$. The vector bundle $F_{\mathsf{c}} \simeq \ker(Tq_A) \cap \ker(p_A)$ is the **core** of F_A.

Recall that the core sections of $TA \to A$ are the sections $a^\uparrow$ as in (2.9) for $a \in \Gamma(A)$, and the core sections of $TA \to TM$ are the sections $\hat{a}$ as in (2.3). The core sections of $TA \to A$ that have image in F_A are hence exactly the core sections $a^\uparrow$ for $a \in \Gamma(F_{\mathsf{c}})$. Now if

$$\hat{a}(v_m) = T_m0^A v_m +_{p_A} \left.\frac{d}{dt}\right|_{t=0} ta(m) \in F_A$$

then we have $v_m = Tq_A(\hat{a}(v_m)) \in F_M(m)$ and since $F_A \to F_M$ is a vector subbundle of $TA \to TM$, the zero element $T_m0^A v_m$ belongs to the fiber of F_A over v_m. This leads to $\left.\frac{d}{dt}\right|_{t=0} ta(m) \in F_A$, using the addition in the fiber of F_A over v_m, and hence $a(m) \in F_{\mathsf{c}}(m)$. *This shows that the core sections $\hat{a}$ of $TA \to TM$ that restrict to core sections of $F_A \to F_M$ are the restrictions $\hat{a}|_{F_M}$ for $a \in \Gamma(F_{\mathsf{c}})$.*

The vector bundles $F_A \to A$ and $F_A \to F_M$ are spanned by their linear and core sections.

Lemma 4.2. *Let (A, F_A) be a foliated Lie algebroid. A section $a \in \Gamma(A)$ is such that $Ta|_{F_M} \in \Gamma_{F_M}(F_A)$ if and only if $\left[a^\uparrow, \Gamma(F_A)\right] \subseteq \Gamma(F_A)$.*

Proof. Choose any linear section X of $F_A \to A$ over a non-vanishing section $\bar{X} \in \Gamma(F_M)$. Since $\bar{X}(m) \neq 0$ for all m, we get that for each $c_m \in A_m$, the vector $X(c_m)$ can be written $X(c_m) = T_m c(\bar{X}(m))$ for some section $c \in \Gamma(A)$ with $c(m) = c_m$. We write for simplicity $\phi^a_\cdot$

for the flow of $a^\uparrow$, i.e., $\phi^a_t(c') = c' + ta(q_A(c'))$ for all $c' \in A$ and $t \in \mathbb{R}$, and $\bar\phi_\cdot$ for the flow of $\bar X$. We have, writing $v_m = \bar X(m)$,

$$\begin{aligned}
T_{c_m}\phi^a_t X(c_m) &= T_m(\phi^a_t \circ c)(\bar X(m))\\
&= \left.\frac{d}{ds}\right|_{s=0} c\left(\bar\phi_s(m)\right) + ta\left(\bar\phi_s(m)\right)\\
&= T_m c(v_m) + tT_m a(v_m)\\
&= X(c_m) + tT_m a(v_m). \qquad (4.15)
\end{aligned}$$

If $\left[a^\uparrow, \Gamma(F_A)\right] \subseteq \Gamma(F_A)$, the flow ϕ^a of $a^\uparrow$ preserves F_A. Hence, $T_{c_m}\phi^a_t X(c_m) \in F_A(c_m + ta(m))$ is a vector of $F_A \to F_M$ over $\bar X(m)$ and since $X(c_m)$ is also a vector of $F_A \to F_M$ over $\bar X(m)$, we find that $tT_m a(v_m) \in F_A(ta(m))$ over $\bar X(m)$ for all $t \in \mathbb{R}$. In particular we get $Ta|_{F_M} \in \Gamma_{F_M}(F_A)$.

Conversely, if $Ta|_{F_M} \in \Gamma_{F_M}(F_A)$, then $tT_m a(v_m) \in F_A(ta(m))$ for all $t \in \mathbb{R}$ and $v_m \in F_M$. Since $X(c_m)$ is an element of F_A for all $c_m \in A_m$, we find hence $T_{c_m}\phi^a_t X(c_m) \in F_A(c_m + ta(m))$ and $[a^\uparrow, X] \in \Gamma(F_A)$. This implies the inclusion $\left[a^\uparrow, \Gamma(F_A)\right] \subseteq \Gamma(F_A)$ since the bracket of two core sections is 0 and $F_A \to A$ is spanned by its linear and its core sections. □

Proposition 4.3. *Let (A, F_A) be a foliated Lie algebroid. Then the core $F_{\mathsf{c}} \to M$ of F_A is a subalgebroid of A and the base $F_M \to M$ is an involutive subbundle of TM.*

Proof. Choose $a, b \in \Gamma(F_{\mathsf{c}})$. Since $F_A \to A$ is involutive and $a^\uparrow \in \Gamma(F_A)$, we know that $\left[a^\uparrow, \Gamma(F_A)\right] \subseteq \Gamma(F_A)$. By the preceding lemma, this implies that $Ta|_{F_M}$ is a section of $F_A \to F_M$. Since $\hat b|_{F_M}$ is also a section of the subalgebroid $F_A \to F_M$ of $TA \to TM$, we find that $\widehat{[a,b]}|_{F_M} = \left[Ta|_{F_M}, \hat b|_{F_M}\right] \in \Gamma_{F_M}(F_A)$ and hence $[a, b] \in \Gamma(F_{\mathsf{c}})$.

The second statement is immediate by Lemma 2.1 and the fact that q_A is a surjective submersion. □

The following lemma will also be useful later.

Lemma 4.4. *Let (A, F_A) be a foliated Lie algebroid. Let $\left(X, 0^{TM}\right)$ be a linear section of TA covering the zero section. Then we have*

$$X \in \Gamma(F_A) \Leftrightarrow D_X a \in \Gamma(F_{\mathsf{c}}) \text{ for any } a \in \Gamma(A).$$

Proof. Choose $a_m \in A$. Since $T_{a_m} q_A(X(a_m)) = 0^{TM}_m$, we can write $X(a_m) = (c(a_m))^\uparrow(a_m)$ for some $c(a_m) \in \Gamma(A)$. We have then, for any section $\varphi \in \Gamma((F_{\mathsf{c}})^\circ)$, where $(F_{\mathsf{c}})^\circ$ is the annihilator of F_{c} in A^*, and any section $a \in \Gamma(A)$ such that $a(m) = a_m$:

$$\begin{aligned}
\langle \varphi(m), D_X a(m)\rangle &= \left.\frac{d}{dt}\right|_{t=0} \langle \varphi(m), \phi^X_{-t}(a_m)\rangle = -X(a_m)(l_\varphi)\\
&= \left.\frac{d}{dt}\right|_{t=0} \langle \varphi(m), a_m - tc(a_m)(m)\rangle = -\langle \varphi(m), c(a_m)(m)\rangle.
\end{aligned}$$

Hence, $(D_X a)(m) \in F_{\mathsf{c}}(m)$ if and only if $c(a_m)(m) \in F_{\mathsf{c}}(m)$. If $D_X a \in \Gamma(F_{\mathsf{c}})$ for all $a \in \Gamma(A)$, we find hence $X \in \Gamma(F_A)$ by (4.14), and conversely, if $X \in \Gamma(F_A)$, we get $D_X a \in \Gamma(F_{\mathsf{c}})$ for all $a \in \Gamma(A)$. □

4.2. **The Lie algebroid of a multiplicative foliation.** The following construction can be found in [26] in the more general setting of multiplicative Dirac structures.

Let F_G be a multiplicative subbundle of TG with space of units $F_M \subseteq TM$. Since $F_G \subseteq TG$ is a Lie subgroupoid, we can apply the Lie functor, leading to a Lie subalgebroid $A(F_G) \subseteq A(TG)$ over $F_M \subseteq TM$.

As we have seen in Subsection 2.2, the canonical involution $J_G : TTG \longrightarrow TTG$ restricts to an isomorphism of double vector bundles $j_G : TA \longrightarrow A(TG)$ inducing the identity map

on both the side bundles and the core. Since $j_G : TA \longrightarrow A(TG)$ is an isomorphism of Lie algebroids over TM, we conclude that

$$F_A := j_G^{-1}(A(F_G)) \subseteq TA$$

is a Lie algebroid over $F_M \subseteq TM$. Since

$$\begin{tikzcd} F_G \arrow[r, "p_G"] \arrow[d, shift right, "T\mathsf{t}"'] \arrow[d, shift left, "T\mathsf{s}"] & G \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] \\ F_M \arrow[r, "p_M"'] & M \end{tikzcd}$$

is a $\mathcal{VB}$-subgroupoid of

$$\begin{tikzcd} TG \arrow[r, "p_G"] \arrow[d, shift right, "T\mathsf{t}"'] \arrow[d, shift left, "T\mathsf{s}"] & G \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] \\ TM \arrow[r, "p_M"'] & M \end{tikzcd},$$

the Lie algebroid

$$\begin{tikzcd} A(F_G) \arrow[r, "A(p_G)"] \arrow[d] & A \arrow[d] \\ F_M \arrow[r, "p_M"'] & M \end{tikzcd}$$

is a $\mathcal{VB}$-subalgebroid of

$$\begin{tikzcd} A(TG) \arrow[r, "A(p_G)"] \arrow[d] & A \arrow[d] \\ TM \arrow[r, "p_M"'] & M \end{tikzcd}$$

([3]), and $F_A \to A$ is also a subbundle of $TA \to A$.

Before we study the sections of $A(F_G) \to F_M$, we show that the vector subbundle $F_A \subseteq TA$ is involutive. The space of sections of $F_A \to A$ is spanned by the core sections $a^\uparrow$ for $a \in \Gamma(F_{\mathsf{c}})$ and linear sections $(\tilde{X}, \bar{X}) = (j_G^{-1}(A(X)), \bar{X})$ coming from star sections $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$ of $F_G \to F_M$. Hence, the involutivity of F_A follows immediately from the involutivity of $F_G \subseteq TG$ and the considerations following Lemma 2.1.

The Lie algebroid $A(F_G) \to F_M$ is a subalgebroid of $A(TG) \to TM$. We want to find the linear sections of $A(TG)$ that restrict to linear sections of $A(F_G)$. If $a \in \Gamma(F_{\mathsf{c}})$, then $\tilde{a}^r$ is easily seen to be tangent to F_G on F_M. Choose an arbitrary section $a \in \Gamma(A)$. We want to find a condition for $\beta_a|_{F_M}$[5] to be a section of $A(F_G)$. We have to find

$$\beta_a(v) \in T_v F_G$$

for all $v \in F_M$. A sufficient condition is $[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G)$. We will show in the remainder of this subsection that this condition is also necessary.

Now we define a more natural, partial F_M-connection ∇^{s} on A/F_{c}, that will help us to understand which linear sections of $A(TG) \to TM$ restrict to linear sections of $A(F_G) \to F_M$. Choose $a \in \Gamma(A)$, $\bar{X} \in \Gamma(F_M)$ and set

$$\nabla^{\mathsf{s}}_{\bar{X}} \bar{a} := \overline{[X, a^r]|_M} \tag{4.16}$$

[5]Recall the definitions of $\tilde{a}$ and β_a from (2.6) and (2.5).

for any section $X \in \Gamma(F_G)$ such that $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$. We show the following proposition.

Proposition 4.5. *Let $(G \rightrightarrows M, F_G)$ be a foliated Lie groupoid. Then ∇^{s} as in (4.16) is a well-defined flat partial F_M-connection on A/F_{c}.*

Proof. For any section $a \in \Gamma(A)$, we have $a^r \sim_{\mathsf{s}} 0$. Since $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$ implies in particular $X \sim_{\mathsf{s}} \bar{X}$, we find $[X, a^r] \sim_{\mathsf{s}} 0$ and $[X, a^r]|_M$ is thus a section of A.

We show that $\overline{[X, a^r]|_M} \in \Gamma(A/F_{\mathsf{c}})$ does not depend on the choice of $X \in \Gamma(F_G)$. If $X' \in \Gamma(F_G)$, $X' \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$, then we have $X - X' \in \Gamma(F_G)$ and $X - X' \overset{\star}{\sim}_{\mathsf{s}} 0$. Hence, if $a_1, \ldots, a_r$ is a local frame for F_{c} on an open set $U \subseteq M$, we can write $X - X' = \sum_{i=1}^r f_i a_i^r$ on $\tilde{U} := \mathsf{t}^{-1}(U)$, with $f_1, \ldots, f_r \in C^\infty(\tilde{U})$ such that $f_i|_U = 0$ for $i = 1, \ldots, r$. We have then

$$[X - X', a^r] = -\sum_{i=1}^r a^r(f_i) a_i^r + \sum_{i=1}^r f_i [a_i, a]^r.$$

But since the second term of this sum vanishes on U, we conclude that $[X - X', a^r]|_M \in \Gamma(F_{\mathsf{c}})$. This proves that ∇^{s} is well-defined.

Assume now that $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$ and choose $f \in C^\infty(M)$. Then we have $(\mathsf{s}^* f) \cdot X \overset{\star}{\sim}_{\mathsf{s}} f \cdot \bar{X}$ and

$$[\mathsf{s}^* f \cdot X, a^r] = \mathsf{s}^* f \cdot [X, a^r] - a^r(\mathsf{s}^* f) X = \mathsf{s}^* f \cdot [X, a^r],$$

which shows

$$\nabla^{\mathsf{s}}_{f \cdot \bar{X}} \bar{a} = f \cdot \nabla^{\mathsf{s}}_{\bar{X}} \bar{a}.$$

Analogously, $(f \cdot a)^r = \mathsf{t}^* f \cdot a^r$ and, since $X|_M = \bar{X} \in \mathfrak{X}(M)$,

$$[X, \mathsf{t}^* f \cdot a^r] = \mathsf{t}^* f \cdot [X, a^r] + X(\mathsf{t}^* f) \cdot a^r$$

restricts to

$$f \cdot [X, a^r]|_M + \bar{X}(f) \cdot a$$

on M.

We show finally that ∇^{s} is flat. Choose $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$ and $Y \overset{\star}{\sim}_{\mathsf{s}} \bar{Y}$. Then, since

$$[X, a^r] - ([X, a^r]|_M)^r$$

vanishes on M and Y is tangent to M on M, we find that

$$\left[Y, [X, a^r] - ([X, a^r]|_M)^r\right]$$

vanishes on M. Thus, we have

$$\left[Y, [X, a^r]\right]|_M = \left[Y, ([X, a^r]|_M)^r\right]|_M$$

and the flatness of ∇^{s} follows easily. □

Recall that we say that $a \in \Gamma(A)$ is ∇- (or ∇^{s}-)parallel if $\bar{a}$ is ∇-parallel, since this doesn't depend on the chosen representative.

Theorem 4.6. *Let $(G \rightrightarrows M, F_G)$ be a foliated Lie groupoid and choose $a \in \Gamma(A)$. Then the following are equivalent:*

(1)
$$[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G),$$
(2) *a is ∇-parallel, where ∇ is the connection in Theorem 3.6,*
(3) *a is ∇^{s}-parallel,*
(4) *$\beta_a|_{F_M}$ is a section of $A(F_G)$,*
(5) *$[X, a^r]|_M \in \Gamma(F_{\mathsf{c}})$ for any section $X \in \Gamma(F_G)$ such that $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$.*

This theorem shows in particular that the ∇- (or equivalently ∇^{s}-) parallel sections of A are *exactly* the sections of A such that $\beta_a|_{F_M}$ is a section of $A(F_G)$.

Since the two connections are flat and their parallel sections coincide, we get the following corollary.

Corollary 4.7. *Let $(G \rightrightarrows M, F_G)$ be a foliated Lie groupoid. Then $\nabla = \nabla^{\mathsf{s}}$.*

Remark 4.8. Note that in Example 3.13, the two connections can immediatly be seen to be the same by definition, since the multiplicative vector fields ξ_G, $\xi \in \mathfrak{h}$ are t-sections and star sections at the same time.

Proof of Theorem 4.6. We already know by Theorem 3.6 that (1) and (2) are equivalent. By definition of ∇^{s}, (3) and (5) are equivalent.

We show that a is ∇-parallel if and only if a is ∇^{s}-parallel. If a is ∇-parallel, then $[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G)$. This yields immediately $[X, a^r]|_M \in \Gamma(F_{\mathsf{c}})$ for any star-section $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$ of F_G. Conversely, assume that a is ∇^{s}-parallel. Since ∇ is flat, a can be written locally as a sum $a = \sum_{i=1}^n f_i a_i$ with $f_i \in C^\infty(M)$, $i = 1, \ldots, n$ and ∇-parallel sections $a_1, \ldots, a_n$. We have then for any section $\bar{X}$ of F_M:

$$\nabla_{\bar{X}} \bar{a} = \sum_{i=1}^n \bar{X}(f_i) \bar{a}_i.$$

But since the sections a_i, $i = 1, \ldots, n$ are then also ∇^{s}-parallel by the considerations above, we have also

$$\sum_{i=1}^n \bar{X}(f_i) \bar{a}_i = \nabla^{\mathsf{s}}_{\bar{X}} \bar{a} = 0,$$

which shows that a is ∇-parallel.

We show that (1) implies (4). If $[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G)$, then the flow $L_{\mathrm{Exp}(\cdot a)}$ of a^r leaves F_G invariant by Corollary A.2. Hence, we have $T_m \operatorname{Exp}(ta) v_m \in F_G(\operatorname{Exp}(ta)(m))$ for all $v_m \in F_M(m)$. This yields $\beta_a(v_m) = \frac{d}{dt}\big|_{t=0} T_m \operatorname{Exp}(ta) v_m \in T^{T\mathsf{s}}_{v_m} F_G = A_{v_m}(F_G)$.

We show that (4) implies (5). Recall that the flow of ${\beta_a}^r$ is equal to $TL_{\mathrm{Exp}(\cdot a)}$. Hence, if $\beta_a|_{F_M} \in A(F_G)$, we have that

$$T_m L_{\mathrm{Exp}(sa)}(v_m) \in F_G(\operatorname{Exp}(sa)(m))$$

for all $|s| \in \mathbb{R}$ small enough and $v_m \in F_M$. We have also

$$T_m L_{\mathrm{Exp}(sa)}(u_m) = 0_{\mathrm{Exp}(sa)(m)} \star u_m \in F_G(\operatorname{Exp}(sa)(m))$$

for all $u_m \in F^{\mathsf{t}}$. This yields

$$T_m L_{\mathrm{Exp}(sa)} F_G(m) \subseteq F_G(\operatorname{Exp}(sa)(m))$$

for all $m \in M$. Since both sides are vector spaces of the same dimension, we get an equality and in particular

$$F_G(m) = T_{\mathrm{Exp}(sa)(m)} L_{\mathrm{Exp}(-sa)} F_G(\operatorname{Exp}(sa)(m)).$$

Choose now any star section $X \overset{\star}{\sim}_{\mathsf{s}} \bar{X}$. We have for any $m \in M$:

$$[a^r, X](m) = \left.\frac{d}{ds}\right|_{s=0} T_{\mathrm{Exp}(sa)(m)} L_{\mathrm{Exp}(-sa)} X(\operatorname{Exp}(sa)(m)).$$

Since $T_{\mathrm{Exp}(sa)(m)} L_{\mathrm{Exp}(-sa)} X(\operatorname{Exp}(sa)(m)) \in F_G(m)$ for all s small enough, we find that $[a^r, X](m) \in F_{\mathsf{c}}(m)$. □

4.3. **Integration of foliated algebroids.** The main theorem of this subsection is the following.

Theorem 4.9. *Let $(G \rightrightarrows M, F_G)$ be a foliated groupoid. Then $(A, F_A = j_G^{-1}(A(F_G)))$ is a foliated algebroid.*

Conversely, let (A, F_A) be a foliated Lie algebroid. Assume that A integrates to a source simply connected Lie groupoid $G \rightrightarrows M$. Then there is a unique multiplicative foliation F_G on G such that $F_A = j_G^{-1}(A(F_G))$.

We will use a result of [3], which states that a $\mathcal{VB}$-algebroid

$$\begin{tikzcd} E \arrow[r, "q_E^h"] \arrow[d, "q_E^v"'] & A \arrow[d, "q_A"] \\ B \arrow[r, "q_B"'] & M \end{tikzcd}$$

integrates to a $\mathcal{VB}$-groupoid

$$\begin{tikzcd} G(E) \arrow[r, "q_{G(E)}"] \arrow[d, shift left] \arrow[d, shift right] & G(A) \arrow[d, shift left] \arrow[d, shift right] \\ B \arrow[r, "q_B"'] & M \end{tikzcd} .$$

Furthermore, if $E' \hookrightarrow E$, $B' \hookrightarrow B$ is a $\mathcal{VB}$-subalgebroid with the same horizontal base A,

$$\begin{tikzcd} E' \arrow[r, "q_E^h"] \arrow[d, "q_E^v"'] & A \arrow[d, "q_A"] \\ B' \arrow[r, "q_{B'}"'] & M \end{tikzcd} ,$$

then $E' \to B'$ integrates to an embedded $\mathcal{VB}$-subgroupoid $G(E') \hookrightarrow G(E)$ over $B' \hookrightarrow B$,

$$\begin{tikzcd} G(E') \arrow[r, "q_{G(E)}"] \arrow[d, shift left] \arrow[d, shift right] & G(A) \arrow[d, shift left] \arrow[d, shift right] \\ B' \arrow[r, "q_M"'] & M \end{tikzcd} .$$

This is done in [3] using the characterization of vector bundles via homogeneous structures (see [10]).

Proof of Theorem 4.9. The first part has been shown in the previous subsection.

Let (A, F_A) be a foliated Lie algebroid. Then the core $F_{\mathsf{c}} \subseteq A$ and the base $F_M \subseteq TM$ are subalgebroids by Proposition 4.3. The $\mathcal{VB}$-subgroupoid of $(TG, G; TM, M)$ integrating the subalgebroid $j_G(F_A) \to F_M$ of $A(TG) \to TM$ is a multiplicative subbundle $F_G \rightrightarrows F_M$ of $TG \rightrightarrows TM$ with core F_{c}. By Theorem 3.10, F_G is involutive. $\square$

4.4. **The connection associated to a foliated algebroid.** Let A be a Lie algebroid endowed with an involutive subbundle $F_A \subseteq TA$. We show that if $F_A \to F_M$ is also morphic, that is, a subalgebroid of the tangent algebroid $TA \to TM$, then the Bott connection

$$\nabla^{F_A} : \Gamma(F_A) \times \Gamma(A/F_A) \to \Gamma(A/F_A)$$

induces a natural partial F_M-connection on A/F_{c} with the same properties as the connection in Theorem 3.6.

For $\bar{X}$ in $\Gamma(F_M)$, we choose any linear section $X \in \Gamma_A(F_A)$ over $\bar{X}$. Then, for any core section $a^{\uparrow}$ of $TA \to A$, we have

$$\left[X, a^{\uparrow}\right] = D_X a^{\uparrow}$$

for some section $D_X a$ of A, see Lemma 2.1. Set

$$\nabla^A_{\bar{X}} \bar{a} = \overline{D_X a} \in \Gamma(A/F_{\mathsf{c}}). \tag{4.17}$$

We show the following proposition.

Proposition 4.10. *Let (A, F_A) be a foliated Lie algebroid. The map*

$$\nabla^A : \Gamma(F_M) \times \Gamma(A/F_{\mathsf{c}}) \to \Gamma(A/F_{\mathsf{c}})$$

as defined in (4.17) *is a well-defined flat partial F_M-connection on A/F_{c}.*

Proof. Choose $\bar{X} \in \Gamma(F_M)$ and two linear sections $X, Y \in \Gamma_A(F_A)$ covering $\bar{X}$. Then $(X - Y, 0)$ is a vector bundle morphism and $Z := X - Y \in \Gamma_A(F_A)$. By Lemma 4.4, we get $D_Z a \in \Gamma(F_{\mathsf{c}})$ for any $a \in \Gamma(A)$ and since

$$(D_Z a)^\uparrow = [X - Y, a^\uparrow] = (D_X a - D_Y a)^\uparrow,$$

this shows that ∇^A is well-defined. The properties of a connection can be checked using the equality $(f \cdot a)^\uparrow = q_A^* f \cdot a^\uparrow$ for $a \in \Gamma(A)$ and $f \in C^\infty(M)$.

The flatness of ∇^A follows immediately from the flatness of the Bott connection ∇^{F_A}. $\square$

Example 4.11. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$. Assume that H acts on a Lie algebroid $A \longrightarrow M$ in a free and proper manner, by Lie algebroid automorphisms. That is, for all $h \in H$, the diffeomorphism Φ_h is a Lie algebroid morphism over $\phi_h : M \to M$. Consider the vertical spaces $\mathcal{V}_A$, $\mathcal{V}_M$ defined as follows

$$\mathcal{V}_A(a) = \{\xi_A(a) \mid \xi \in \mathfrak{h}\}, \qquad \mathcal{V}_M(m) = \{\xi_M(m) \mid \xi \in \mathfrak{h}\},$$

for $a \in A$ and $m \in M$. We check that $\mathcal{V}_A$ inherits a Lie algebroid structure over $\mathcal{V}_M$ making the pair $(A, \mathcal{V}_A)$ into a foliated algebroid with core zero. Choose $\xi_A(a_m) \in \mathcal{V}_A(a_m)$ for some $\xi \in \mathfrak{g}$, then $T_{a_m} q_A \xi_A(a_m) = \xi_M(m) \in \mathcal{V}_M(m)$. Hence, if $T_{a_m} q_A \xi_A(a_m) = 0$, then $\xi = 0$ since the action is supposed to be free and we find $F_{\mathsf{c}} = 0$. Notice that, since the action is by algebroid automorphisms, the infinitesimal generators ξ_A are in fact morphic vector fields covering ξ_M.

We have, writing $\rho(a_m) = \dot{c}(0)$,

$$\begin{aligned}\rho_{TA}(\xi_A(a_m)) &= J_M\left(T\rho(\xi_A(a_m)) = J_M\left(\left.\frac{d}{dt}\right|_{t=0} (\rho \circ \Phi_{\exp(t\xi)})(a_m)\right)\right. \\ &= J_M\left(\left.\frac{d}{dt}\right|_{t=0} (T\phi_{\exp(t\xi)} \circ \rho)(a_m)\right) = \left.\frac{d}{ds}\right|_{s=0} \xi_M(c(s)) \in T_{\xi_M(m)}\mathcal{V}_M.\end{aligned}$$

If $a \in \Gamma(A)$ is such that $T_m a(\xi_M(m)) \in \mathcal{V}_A(a_m)$ for some $\xi \in \mathfrak{g}$ and $m \in M$, then there exists $\eta \in \mathfrak{g}$ such that $T_m a(\xi_M(m)) = \eta_A(a(m))$. But applying $T_{a_m} q_A$ to both sides of this equality yields then $\xi_M(m) = \eta_M(m)$, which leads to $\xi = \eta$, since the action is free, and hence $T_m a(\xi_M(m)) = \xi_A(a(m))$.

Here, the induced partial $\mathcal{V}_M$-connection on A is given by $\nabla^A_{\xi_M} a \in \Gamma(A)$ where

$$[\xi_A, a^\uparrow] = (\nabla^A_{\xi_M} a)^\uparrow$$

for any $a \in \Gamma(A)$. If $a \in \Gamma(A)$ is ∇-parallel, then we find $[\xi_A, a^\uparrow] = 0$ for all $\xi \in \mathfrak{g}$ and hence

$$\begin{aligned}0 &= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \Phi_{\exp(s\xi)}(b_m + ta(m)) - ta(\phi_{\exp(s\xi)}) \\ &= \left.\frac{d}{dt}\right|_{t=0} \xi_A(b_m) + t \cdot \left.\frac{d}{ds}\right|_{s=0} \left(\Phi_{\exp(s\xi)}(a(m)) - a(\phi_{\exp(s\xi)})\right)\end{aligned}$$

for all $b_m \in A$. This leads to $\xi_A(a(m)) = T_m a \xi_M(m)$ and hence

$$Ta(\mathcal{V}_M) \subseteq \mathcal{V}_A. \tag{4.18}$$

The parallel sections of ∇^A are hence exactly the sections of A satisfying (4.18). This will be shown in the general situation in the next lemma.

If $a \in \Gamma(A)$ is ∇^A-parallel, we find easily that $a(\phi_h(m)) = \Phi_h(a(m))$ for all $h \in H$ (recall that H is assumed to be connected) and $m \in M$. Hence, if $a, b \in \Gamma(A)$ are such that

$Ta(\mathcal{V}_M) \subseteq \mathcal{V}_A$ and $Tb(\mathcal{V}_M) \subseteq \mathcal{V}_A$, we have $a \circ \phi_h = \Phi_h \circ a$ and $b \circ \phi_h = \Phi_h \circ b$ for all $h \in H$. Since the action is by Lie algebroid morphisms, we get then $[a,b] \circ \phi_h = \Phi_h \circ [a,b]$, which implies $T[a,b](\mathcal{V}_M) \subseteq \mathcal{V}_A$. Since the core sections of $\mathcal{V}_A$ are all trivial, this shows that the Lie bracket of $TA \to TM$ restricts to $\mathcal{V}_A \to \mathcal{V}_M$.

Lemma 4.12. *Let (A, F_A) be a foliated Lie algebroid and ∇^A the connection defined in*(4.17)*. The following are equivalent for $a \in \Gamma(A)$:*

1. *the class $\bar{a}$ is ∇^A-parallel,*
2. *$Ta|_{F_M}$ is a section of F_A,*
3. *the core section $a^\uparrow$ of $F_A \to A$ is parallel with respect to the Bott connection ∇^{F_A}, that is $\left[a^\uparrow, \Gamma(F_A)\right] \subseteq \Gamma(F_A)$.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) has been shown in Lemma 4.2 and the implication (3) $\Rightarrow$ (1) is obvious.

If $\nabla^A_{\bar{X}}\bar{a} = 0$ for all $\bar{X} \in \Gamma(F_M)$, we get that

$$\left[X, a^\uparrow\right] \in \Gamma(F_A)$$

for all linear sections $X \in \Gamma(F_A)$. Since

$$\left[a^\uparrow, b^\uparrow\right] = 0 \in \Gamma(F_A)$$

for all $b \in \Gamma(F_{\mathsf{c}})$ and the linear sections of F_A together with the core sections of F_A span F_A, we get

$$\left[a^\uparrow, \Gamma(F_A)\right] \subseteq \Gamma(F_A)$$

and the flow of $a^\uparrow$ leaves F_A invariant.

Hence, we have shown (1) $\Rightarrow$ (3). □

Now we can complete the proof of the main theorem of this subsection.

Theorem 4.13. *Let (A, F_A) be a foliated Lie algebroid. Then the partial F_M-connection ∇^A on A/F_{c} as in* (4.17) *has the same properties as the connection ∇ in Theorem 3.6, and such that*

$$\bar{a} \text{ is } \nabla^A\text{-parallel} \quad \Leftrightarrow \quad Ta(F_M) \subseteq F_A.$$

Proof. By Proposition 4.10, we know that ∇^A as in (4.17) is well-defined and flat and we have shown in Lemma 4.12 that $\bar{a}$ is ∇^A-parallel if and only if $Ta(F_M) \subseteq F_A$.

If $c \in \Gamma(F_{\mathsf{c}})$, then $\hat{c}|_{F_M}$ is a section of $F_A \to F_M$, and $\widehat{[a,c]}|_{F_M} = [Ta|_{F_M}, \hat{c}|_{F_M}] \in \Gamma_{F_M}(F_A)$. Hence, $[a,c] \in \Gamma(F_{\mathsf{c}})$.

If $b \in \Gamma(A)$ is also such that $\bar{b}$ is ∇^A-parallel, and $Ta|_{F_M}, Tb|_{F_M} \in \Gamma_{F_M}(F_A)$, then, since $F_A \to F_M$ is a subalgebroid of $TA \to TM$, we find that $[Ta|_{F_M}, Tb|_{F_M}] \in \Gamma_{F_M}(F_A)$. But since $[Ta|_{F_M}, Tb|_{F_M}] = T[a,b]|_{F_M}$, this shows that $\overline{[a,b]}$ is ∇^A-parallel by Lemma 4.12.

It remains to show that $\rho(a) \in \mathfrak{X}(M)$ is ∇^{F_M}-parallel if a is ∇^A-parallel. Since $Ta|_{F_M} \in \Gamma_{F_M}(F_A)$ and $F_A \to F_M$ is a subalgebroid of $TA \to TM$, we find that

$$\rho_{TA}(T_m a(v_m)) = J_M(T_m(\rho_A(a))v_m)$$

is an element of $T_{v_m}F_M$ for any $v_m \in F_M$. Choose $X \in \Gamma(F_M)$ and $\alpha \in \Gamma(F_M^\circ)$. Then, if $l_\alpha \in C^\infty(TM)$ is the linear function defined by α, we have

$$\mathbf{d}_{X(m)} l_\alpha(\rho_{TA}(T_m a(X(m)))) = 0.$$

But the identity

$$J_M(T_m(\rho_A(a))X(m)) = \left.\frac{d}{dt}\right|_{t=0} T_m\phi_t X(m),$$

where $\phi_\cdot$ is the flow of $\rho_A(a)$, yields then immediately

$$\left(\pounds_{\rho_A(a)}\alpha\right)(X(m)) = 0$$

for all m, and the equality

$$0 = \pounds_{\rho_A(a)}(\alpha(X)) = (\pounds_{\rho_A(\alpha)}\alpha)(X) + \alpha([\rho_A(a), X])$$

leads thus to $\alpha([\rho_A(a), X]) = 0$. Since α and X were arbitrary sections of F_M° and F_M, respectively, we have shown that $[\rho_A(a), \Gamma(F_M)] \subseteq \Gamma(F_M)$. □

Theorem 4.14. *Let (A, F_A) be a foliated algebroid. Assume that A integrates to a Lie groupoid $G \rightrightarrows M$ and that $j_G(F_A)$ integrates to $F_G \subseteq TG$. Then*

$$\nabla = \nabla^{\mathsf{s}} = \nabla^A.$$

Proof. It is clear that if $j_G(F_A)$ integrates to F_G, then the core and the base of F_G and of F_A coincide. Hence, the three connections are flat partial F_M-connections on A/F_{c}. Since $\nabla = \nabla^{\mathsf{s}}$ by Corollary 4.7, we only need to show that ∇ and ∇^A have the same parallel sections. For that we use Theorem 4.6. The class $\bar{a}$ of $a \in \Gamma(A)$ is ∇-parallel if and only if $\beta_a|_{F_M}$ is a section of $A(F_G)$. Since $F_A = j_G^{-1}(A(F_G))$ and $j_G^{-1} \circ \beta_a|_{F_M} = Ta|_{F_M}$, this is equivalent to $Ta|_{F_M} \in \Gamma_{F_M}(F_A)$. But by the preceding theorem, this is true if and only if $\bar{a}$ is ∇^A-parallel. □

5. The connection as the infinitesimal data of a multiplicative foliation

5.1. **Infinitesimal description of a multiplicative foliation.** We have seen that both foliated groupoids and foliated algebroids induce partial connections which have important properties. This motivates the following concept.

Definition 5.1. *Let $(A \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid, $F_M \subseteq TM$ an involutive subbundle, $F_{\mathsf{c}} \subseteq A$ a subalgebroid over M such that $\rho(F_{\mathsf{c}}) \subseteq F_M$ and ∇ a partial F_M-connection on A/F_{c} with the following properties:*

1. *∇ is flat.*
2. *If $a \in \Gamma(A)$ is ∇-parallel, then $[a, b] \in \Gamma(F_{\mathsf{c}})$ for all $b \in \Gamma(F_{\mathsf{c}})$.*
3. *If $a, b \in \Gamma(A)$ are ∇-parallel, then $[a, b]$ is also parallel.*
4. *If $a \in \Gamma(A)$ is ∇-parallel, then $[\rho(a), \bar{X}] \in \Gamma(F_M)$ for all $\bar{X} \in \Gamma(F_M)$. That is, $\rho(a)$ is ∇^{F_M}-parallel.*

The quadruple $(A, F_M, F_{\mathsf{c}}, \nabla)$ will be referred to as an ***IM-foliation***[6] ***on*** *A, or simply an* ***IM-foliation****.*

Remark 5.2.
1. IM-foliations already appear in the work of Eli Hawkins [11], where they are also found to correspond with foliated algebroids.
2. Note that this is an infinitesimal version of the **ideal systems** in [21].

We have shown in Theorem 3.6 that each foliated Lie groupoid $(G \rightrightarrows M, F_G)$ induces an IM-foliation on A. In the same manner, we have seen in Theorem 4.13 that each foliated Lie algebroid (A, F_A) induces an IM-foliation on A. Moreover, a foliated Lie groupoid $(G \rightrightarrows M, F_G)$ and its foliated Lie algebroid $(A, j_G^{-1}(A(F_G)))$ induce the same IM-foliation on A.

We will show here that an IM-foliation on a Lie algebroid is equivalent to a morphic foliation on this Lie algebroid. *This implies that IM-foliations on integrable Lie algebroids are in one-to-one correspondence with multiplicative foliations on the corresponding Lie groupoids.*

[6]Here, "IM" stands for "infinitesimal multiplicative", as for the IM-2-forms in [5].

5.2. **Reconstruction of the foliated Lie algebroid from the connection.** Let now $(q_A : A \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid and $(A, F_M, F_{\mathsf{c}}, \nabla)$ an IM-foliation on A.

The Lie algebroid $TA \to TM$ is spanned by the sections Ta and $\hat{a}$ for all $a \in \Gamma(A)$. Recall that $\hat{a}(v_m) = T_m 0^A v_m +_{p_A} \frac{d}{dt}\big|_{t=0} ta(m)$ for $v_m \in T_mM$ and that the bracket of the Lie algebroid $TA \to TM$ is given by

$$[Ta, Tb]_{TA} = T[a,b], \qquad \left[Ta, \hat{b}\right]_{TA} = \widehat{[a,b]}, \qquad \left[\hat{a}, \hat{b}\right]_{TA} = 0$$

for all $a, b \in \Gamma(A)$.

For each $v_m \in F_M$, define the subset $F_A(v_m)$ of $(TA)_{v_m}$ as follows:

$$F_A(v_m) = \operatorname{span}\left\{ \begin{array}{c|c} T_m a(v_m) \in T_{a(m)}A, & a \in \Gamma(A) \text{ is } \nabla\text{-parallel}, \\ & b \in \Gamma(F_{\mathsf{c}}) \\ \hat{b}(v_m) & \end{array} \right\}. \tag{5.19}$$

That is, F_A is spanned by the restrictions to F_M of the linear sections Ta of TA defined by $a \in \Gamma(A)$ such that $\bar{a} \in \Gamma(A/F_{\mathsf{c}})$ is ∇-parallel and the core sections $\hat{b}$ defined by sections $b \in \Gamma(F_{\mathsf{c}})$. We will show that F_A is a well-defined subalgebroid of $TA \to TM$.

First, we check that F_A is a double vector bundle

$$\begin{tikzcd} F_A \arrow[r] \arrow[d] & A. \arrow[d] \\ F_M \arrow[r] & M \end{tikzcd}$$

By construction, F_A has core F_{c} and there will be an injective double vector bundle morphism to

$$\begin{tikzcd} TA \arrow[r, "p_A"] \arrow[d, "Tq_A"'] & A \arrow[d, "q_A"] \\ TM \arrow[r, "p_M"'] & M \end{tikzcd}$$

which has core A.

By definition, we have $T_{a_m} q_A(T_m a(v_m)) = T_m(q_A \circ a)v_m = v_m \in F_M(m)$ for all $a \in \Gamma(A)$ and $v_m \in F_M(m)$ and $T_{v_m} q_A\left(\hat{b}(v_m)\right) = v_m \in F_M(m)$ for all $b \in \Gamma(F_{\mathsf{c}})$.

Choose $m \in M$. Then there exists a neighborhood U of m in M and sections $a_1, \ldots, a_n \in \Gamma(A)$ such that $\overline{a_{r+1}}, \ldots, \overline{a_n}$ are ∇-parallel and form a basis for A/F_{c} on U, and $a_1, \ldots, a_r$ form a basis for F_{c} on U. Note that $\overline{a_1}, \ldots, \overline{a_r}$ are in a trivial manner ∇-parallel. Choose any ∇-parallel section $a \in \Gamma(A)$. Then, on U, we have $a = \sum_{i=1}^n f_i a_i$ with $f_1, \ldots, f_r \in C^\infty(U)$ and F_M-invariant functions $f_{r+1}, \ldots, f_n \in C^\infty(U)$. For any $v_m \in F_M(m)$, the equality

$$T_m a(v_m) = \sum_{i=1}^r v_m(f_i)\widehat{a_i}(v_m) + \sum_{i=1}^n f_i(m) T_m a_i(v_m)$$

can be checked in coordinates, using $v_m(f_i) = 0$ for $i = r+1, \ldots, n$. If

$$\sum_{i=1}^n \alpha_i T_m a_i(v_m) + \sum_{i=1}^r \beta_i \widehat{a_i}(v_m) = 0^{TA}_{v_m}$$

for some $v_m \in F_m$ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_r \in \mathbb{R}$, we find

$$\sum_{i=1}^n \alpha_i a_i(m) = p_A\left(\sum_{i=1}^n \alpha_i T_m a_i(v_m) + \sum_{i=1}^r \beta_i \widehat{a_i}(v_m)\right) = 0^A_m.$$

Since $a_1, \ldots, a_n$ is a local frame for A, this yields $\alpha_1 = \ldots = \alpha_n = 0$ and the equality $\beta_1 = \ldots = \beta_r = 0$ follows easily. The set of sections $Ta_1, \ldots, Ta_n$ and $\widehat{a_1}, \ldots, \widehat{a_r}$ is thus a local frame for F_A as a vector bundle over F_M.

We next show that F_A is also a vector bundle over A (a subbundle of $TA \to A$). Choose $a_m \in A_m$. Then there exists a ∇-parallel section $a \in \Gamma(A)$ defined on a neighborhood U of m such that $a(m) = a_m$. Let r be the rank of F_{c} and l the rank of F_M. Choose a basis of vector fields $V_1, \ldots, V_l$ for F_M on U and a basis of sections $b_1, \ldots, b_r$ for F_{c} on U. Choose a local frame $a_1, \ldots, a_n$ for A on U by ∇-parallel sections and define for $i = 1, \ldots, l$ the sections $\tilde{V}_i : q_A^{-1}(U) \to F_A$ by

$$\tilde{V}_i(a_m) = T\left(\phi_{\alpha_1}^{a_1^\uparrow} \circ \ldots \circ \phi_{\alpha_l}^{a_l^\uparrow} \circ 0^A\right)(V_i(m))$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ are such that $a_m = \sum_{i=1}^n \alpha_i \cdot a_i(m)$. By construction, the vector fields $\tilde{V}_i$ are sections of $F_A \to A$ (recall Lemma 4.12), and, using the fact that core vector fields commute, it is easy to show that $\tilde{V}_i$ is linear over V_i for each $i = 1, \ldots, l$. The sections $\tilde{V}_1, \ldots, \tilde{V}_l, b_1^\uparrow, \ldots, b_r^\uparrow$ form a basis for F_A (seen as vector bundle over A) on $q_A^{-1}(U)$.

Now we check that $(F_A, \rho_{TA}, [\cdot\,,\cdot]_{TA})$ is a Lie algebroid over F_M (a subalgebroid of $TA \to TM$). Choose two linear sections $Ta|_{F_M}$ and $Tb|_{F_M}$ of F_A, i.e., with $\bar{a}, \bar{b} \in \Gamma(A/F_{\mathsf{c}})$ that are ∇-parallel. We have then $[Ta, Tb]_{TA} = T[a,b]$. By the properties of the connection, $[a,b]$ is ∇-parallel and $T[a,b]|_{F_M}$ is a section of F_A. Choose now a core section $\hat{b}|_{F_M}$ and a linear section $Ta|_{F_M}$ of F_A, i.e., with $b \in \Gamma(F_{\mathsf{c}})$ and $a \in \Gamma(A)$ a ∇-parallel section. We get $[Ta, \hat{b}]_{TA} = \widehat{[a,b]}$, whose restriction to F_M is a section of F_A since, by the properties of the connection, $[a,b] \in \Gamma(F_{\mathsf{c}})$. Since the bracket of two core sections vanishes, we have shown that $[\cdot\,,\cdot]_{TA}$ restricts to F_A.

Next, we show that the anchor map of TA restricts to a map $F_A \to TF_M$. Again, it is sufficient to show that the spanning sections of F_A are sent by ρ_{TA} to vector fields on F_M. Recall that the anchor map is given by $\rho_{TA} = J_M \circ T\rho$. Choose a ∇-parallel section $a \in \Gamma(A)$, set $X := \rho(a) \in \mathfrak{X}(M)$ and compute for any $v_m = \dot{c}(0) \in F_M$:

$$\rho_{TA}(T_m a(v_m)) = J_M(T_m(\rho(a))v_m) = \left.\frac{d}{ds}\right|_{s=0} T_m\phi_s^X(v_m).$$

By the properties of the connection, the vector field X is such that $[X, \Gamma(F_M)] \subseteq \Gamma(F_M)$. By Corollary A.2, this implies $T_m\phi_s^X F_M(m) = F_M(\phi_s^X(m))$ for all $m \in M$ and $s \in \mathbb{R}$ where this makes sense. This implies that the curve $s \mapsto T_m\phi_s^X(v_m)$ has image in F_M and consequently, $\rho_{TA}(T_m a(v_m)) \in T_{v_m}F_M$.

Similarly, choose $b \in \Gamma(F_{\mathsf{c}})$, $v_m = \dot{c}(0) \in F_M(m)$, set $Y = \rho(b) \in \Gamma(F_M)$ and compute:

$$\begin{aligned}
&\rho_{TA}\left(T_m 0^A(v_m) +_{p_A} \left.\frac{d}{dt}\right|_{t=0} tb(m)\right)\\
&= J_M\left(T_{0_m}\rho\left(T_m 0^A(v_m) +_{p_A} \left.\frac{d}{dt}\right|_{t=0} tb(m)\right)\right)\\
&= J_M\left(\left.\frac{d}{dt}\right|_{t=0} 0^{TM}(c(t)) +_{p_{TM}} \left.\frac{d}{dt}\right|_{t=0} tY(m)\right)\\
&= J_M\left(\left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} c(t) +_{p_{TM}} \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \phi_{st}^Y(m)\right)\\
&= \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} c(t) +_{Tp_M} \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} \phi_{st}^Y(m)\\
&= \left.\frac{d}{ds}\right|_{s=0} v_m +_{TM} sY(m) \in T_{v_m}F_M.
\end{aligned}$$

Note also that the induced map

$$\begin{tikzcd} F_A \arrow[r, "p_A"] \arrow[d, "Tq_A"'] & A \arrow[d, "q_A"] \\ F_M \arrow[r, "p_M"'] & M \end{tikzcd}$$

is a surjective Lie algebroid morphism by construction.

We have proved the following proposition.

Proposition 5.3. *Let $(A \to M, F_M, F_{\mathsf{c}}, \nabla)$ be an IM-foliation on A. Consider $F_A \subseteq TA$ constructed as in* (5.19). *Then $F_A \to F_M$ is a subalgebroid of $TA \to TM$.*

We show now that $F_A \to A$ is also a subalgebroid of $TA \to A$.

Proposition 5.4. *Let $(A \to M, F_M, F_{\mathsf{c}}, \nabla)$ be an IM-foliation on A. Consider $F_A \to F_M$ the Lie subalgebroid of $TA \to TM$ as above. The vector subbundle $F_A \to A$ of $TA \to A$ is involutive.*

Proof. By definition, the linear sections of $TA \to TM$ that restrict to sections of $F_A \to F_M$ are the sections Ta for all $a \in \Gamma(A)$ such that $\bar{a}$ is a ∇-parallel section of A/F_{c}. Note also that, by definition, every section of F_{c} is ∇-parallel in this sense.

As in the proof of Lemma 4.2, this implies $[a^\uparrow, \Gamma(F_A)] \subseteq \Gamma(F_A)$ if $a \in \Gamma(A)$ is a ∇-parallel section. As a consequence, for any linear section X of F_A over $\bar{X}$, we have $[X, a^\uparrow] = (D_X a)^\uparrow$ with $D_X a \in \Gamma(F_{\mathsf{c}})$. We find thus the trivial equality $\nabla_{\bar{X}}\bar{a} = \overline{D_X a}$ for ∇-parallel sections $a \in \Gamma(A)$.

Since ∇ is flat, there exist local frames for A of ∇-parallel sections. Choose $a \in \Gamma(A)$ and write $a = \sum_{i=1}^n f_i a_i$ with $a_1, \ldots, a_n \in \Gamma(A)$ ∇-parallel sections and $f_1, \ldots, f_n \in C^\infty(M)$. Then $\nabla_{\bar{X}}\bar{a} = \sum_{i=1}^n \bar{X}(f_i)\bar{a}_i$. But, on the other hand, if $X \in \Gamma(F_A)$ is a linear section over $\bar{X}$, we have

$$(D_X a)^\uparrow = \left[X, \left(\sum_{i=1}^n f_i a_i\right)^\uparrow\right] = \sum_{i=1}^n q_A^*\left(\bar{X}(f_i)\right) a_i^\uparrow + \sum_{i=1}^n q_A^* f_i (D_X a_i)^\uparrow,$$

which leads to

$$\nabla_{\bar{X}}\bar{a} = \sum_{i=1}^n \bar{X}(f_i)\bar{a}_i = \sum_{i=1}^n \bar{X}(f_i)\bar{a}_i + \sum_{i=1}^n f_i \cdot \overline{D_X a_i} = \overline{D_X a},$$

since $D_X a_i \in \Gamma(F_{\mathsf{c}})$ for $i = 1, \ldots, n$.

Choose now two linear sections $(X, \bar{X})$ and $(Y, \bar{Y})$ of $F_A \to A$. Since ∇ is flat and F_M is involutive , we have $\nabla_{[\bar{X},\bar{Y}]}\bar{a} = \nabla_{\bar{X}}(\nabla_{\bar{Y}}\bar{a}) - \nabla_{\bar{Y}}(\nabla_{\bar{X}}\bar{a})$ for all $a \in \Gamma(A)$. Hence, if $\nabla_{[\bar{X},\bar{Y}]}\bar{a} = \bar{k}$ for some $k \in \Gamma(A)$, then

$$(k+b)^\uparrow = \left[X, \left[Y, a^\uparrow\right]\right] - \left[Y, \left[X, a^\uparrow\right]\right] = \left[[X,Y], a^\uparrow\right]$$

for some $b \in \Gamma(F_{\mathsf{c}})$ and, if $Z \sim_{q_A} [\bar{X}, \bar{Y}]$ is a linear section of F_A covering $[\bar{X}, \bar{Y}]$, then

$$(k+b')^\uparrow = \left[Z, a^\uparrow\right]$$

for some $b' \in \Gamma(F_{\mathsf{c}})$. The linear section $W := Z - [X, Y]$ covers hence 0^{TM} and satisfies

$$\left[W, a^\uparrow\right] = (b - b')^\uparrow.$$

Since $b - b' \in \Gamma(F_{\mathsf{c}})$, we have found $D_W a \in \Gamma(F_{\mathsf{c}})$ for any $a \in \Gamma(A)$, and, combining this with Lemma 4.4, one concludes that $W \in \Gamma(F_A)$ and hence $[X, Y] \in \Gamma(F_A)$. Since $F_A \to A$ is spanned by its linear and core sections, we have shown that F_A is involutive. □

This completes the proof of our main theorem.

Theorem 5.5. *Let $(q_A : A \to M, \rho_A, [\cdot\,,\cdot]_A)$ be a Lie algebroid and $(A, F_M, F_{\mathsf{c}}, \nabla)$ an IM-foliation on A. Then F_A defined as in (5.19) is a $\mathcal{VB}$-algebroid with sides F_M and A and core F_{c}, such that the inclusion is a $\mathcal{VB}$-algebroid morphism:*

$$\begin{tikzcd}
F_A \arrow[rr] \arrow[dd] \arrow[dr, hook] & & A \arrow[dd] \arrow[dr, equal] & \\
& TA \arrow[rr] \arrow[dd] & & A \arrow[dd] \\
F_M \arrow[rr] \arrow[dr, hook] & & M \arrow[dr, equal] & \\
& TM \arrow[rr] & & M
\end{tikzcd}$$

Hence, (A, F_A) is a foliated Lie algebroid and the connection ∇^A induced by F_A is equal to ∇.

Example 5.6. Assume that H acts on a Lie groupoid G over M by groupoid automorphisms. Assume also that the action is free and proper. Starting from the data $(A, \mathcal{V}_M, 0, \nabla)$ where ∇ is the partial $\mathcal{V}_M$-connection on A determined by

$$[\xi_G, a^r] = (\nabla_{\xi_M} a)^r$$

for all $\xi \in \mathfrak{g}$ and $a \in \Gamma(A)$, the last theorem states that we recover exactly the Lie foliated algebroid $\mathcal{V}_A \longrightarrow \mathcal{V}_M$ obtained by applying the Lie functor to the foliated groupoid $\mathcal{V}_G \rightrightarrows \mathcal{V}_M$.

Example 5.7. Assume that $\mathfrak{g}$ is a Lie algebra, i.e. a Lie algebroid over a point. In this case, the tangent Lie algebroid $T\mathfrak{g}$ is also a Lie algebroid over a point, that is, $T\mathfrak{g}$ is a Lie algebra. It is easy to see that the Lie algebra structure on $T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ is the semi-direct product Lie algebra $\mathfrak{g} \ltimes \mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on itself. Note also that the fact that a quadruple $(\mathfrak{g}, 0, \mathfrak{f}, \nabla = 0)$ is an IM-foliation on $\mathfrak{g}$ is equivalent to saying that $\mathfrak{f} \subseteq \mathfrak{g}$ is an ideal.

The morphic foliation $F_{\mathfrak{g}}$ associated to the IM-foliation $(\mathfrak{g}, 0, \mathfrak{f}, \nabla = 0)$ is given by $F_{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{f}$. This follows immediately from the fact that every element $a \in \mathfrak{g}$ can be viewed, in a trivial way, as a ∇-parallel section of $\mathfrak{g} \to \{0\}$. The property that $F_{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{f}$ is a morphic foliation is equivalent to saying that $\mathfrak{g} \times \mathfrak{f}$ is a Lie subalgebra of $\mathfrak{g} \ltimes \mathfrak{g}$. In particular, if G is the connected and simply connected Lie group integrating $\mathfrak{g}$, we conclude that the foliated algebroid $F_{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{f}$ integrates to a Lie subgroup $G \times \mathfrak{f}$ of the semi-direct Lie group $G \ltimes \mathfrak{g}$ determined by the adjoint action of G on its Lie algebra $\mathfrak{g}$. Using right (or left) translations, we get a subbundle $F_G \subseteq TG$ which is involutive and multiplicative. We conclude that F_G is the multiplicative foliation on G integrating the IM-foliation $(\mathfrak{g}, 0, \mathfrak{f}, \nabla = 0)$ on $\mathfrak{g}$. Thus, in the case of Lie groups and Lie algebras, this recovers the results in [25, 14, 15].

6. The leaf space of a foliated algebroid

Assume that $(A, F_M, F_{\mathsf{c}}, \nabla)$ is an IM-foliation on A. Then there is an induced involutive subbundle $F_A \subseteq TA$ as in Theorem 5.5. We will show that if the leaf space M/F_M is a smooth manifold, such that the quotient map $\pi_M : M \to M/F_M$ is a surjective submersion, then there is an induced Lie algebroid structure $([q_A] : A/F_A \to M/F_M, [\rho], [\cdot\,,\cdot]_{A/F_A})$ such

that the projection

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "q_A"'] & A/F_A \arrow[d, "{[q_A]}"] \\ M \arrow[r, "\pi_M"'] & M/F_M \end{tikzcd}$$

is a Lie algebroid morphism. Furthermore, if $A \to M$ integrates to a Lie groupoid $G \rightrightarrows M$ and the completeness and regularity conditions for the leaf space $G/F_G \rightrightarrows M/F_M$ to be a Lie groupoid are satisfied (see [15]), then $A/F_A \to M/F_M$ is the Lie algebroid of $G/F_G \rightrightarrows M/F_M$. We will see in the proofs that the important data for this reduction process is the IM-foliation $(A, F_M, F_{\mathsf{c}}, \nabla)$.

The class of $a_m \in A_m$ is written $[a_m] \in A/F_A$, and in the same manner, the class of $m \in M$ is denoted by $[m] \in M/F_M$. The class of $a_m \in A_m$ in A/F_{c} will be written $\bar{a}_m$.

Proposition 6.1. *Let $(A \to M, F_M, F_{\mathsf{c}}, \nabla)$ be an IM-foliation on A and $F_A \subseteq TA$ the corresponding morphic foliation as in Theorem 5.5.*

(1) *The map $\pi : A \to A/F_A$ factors as a composition*

$$\begin{tikzcd} A \arrow[d] \arrow[dr, "\pi"] & \\ A/F_{\mathsf{c}} \arrow[r, "\pi_{\mathsf{c}}"'] & A/F_A \end{tikzcd}$$

That is, we have $\pi(a_m + b_m) = \pi(a_m)$ for all $a_m \in A$ and $b_m \in F_{\mathsf{c}}(m)$.

(2) *The equivalence relation $\sim := \sim_{F_A}$ on A can be described as follows.*

$$a_m \sim a_n \Leftrightarrow \begin{array}{c} \textit{There exist linear sections } (X_1, \bar{X}_1), \ldots, (X_r, \bar{X}_r) \textit{ of } F_A \to A \\ \textit{with flows } \phi^1, \ldots, \phi^r \textit{ such that} \\ a_m \in \phi^1_{t_1} \circ \ldots \circ \phi^r_{t_r}(a_n) + F_{\mathsf{c}}(m) \\ \textit{for some } t_1, \ldots, t_r \in \mathbb{R}. \end{array} \tag{6.20}$$

(3) *The map q_A induces a map $[q_A] : A/F_A \to M/F_M$ such that*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "q_A"'] & A/F_A \arrow[d, "{[q_A]}"] \\ M \arrow[r, "\pi_M"'] & M/F_M \end{tikzcd}$$

commutes.

(4) *Let $a \in \Gamma(A)$ be such that $\bar{a} \in \Gamma(A/F_{\mathsf{c}})$ is ∇-parallel. Let $U := \mathrm{Dom}(a) \subseteq M$. Then there is an induced map $[a] : \pi_M(U) \to A/F_A$ such that*

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "\pi_M"'] & A \arrow[d, "\pi"] \\ \pi_M(U) \arrow[r, "{[a]}"'] & A/F_A \end{tikzcd}$$

commutes. The map $[a]$ is a section of $[q_A]$ in the sense that

$$[q_A] \circ [a] = \mathrm{Id}_{\pi_M(U)}.$$

Proof. (1) Recall that all the core sections $b^\uparrow \in \mathfrak{X}(A)$ with $b \in \Gamma(F_{\mathsf{c}})$ are sections of F_A. Choose $a_m \in A$ and $b_m \in F_{\mathsf{c}}(m)$. Then there exists a section $b \in \Gamma(F_{\mathsf{c}})$ with $b(m) = b_m$. The flow $\phi^{b^\uparrow}$ of $b^\uparrow$ is given by $\phi^{b^\uparrow}_t(a) = a + tb(q_A(a))$ for all $a \in A$ and $t \in \mathbb{R}$. Hence, we have $a_m \sim a_m + tb(m) = a_m + tb_m$ for all $t \in \mathbb{R}$, and in particular

$a_m \sim a_m + b_m$. The map $\pi_{\mathsf{c}} : A/F_{\mathsf{c}} \to A/F_A$, $\pi_{\mathsf{c}}(a_m + F_{\mathsf{c}}(m)) = [a_m]$ is hence well-defined and the diagram commutes.

(2) Since the family of linear sections of F_A and the family of core sections of F_A span together F_A, its leaves are the accessible sets of these two families of vector fields ([28, 29, 30], see [24] for a review of these results). Hence, two points a_m and a_n in A are in the same leaf of F_A if they can be joined by finitely many curves along flow lines of core sections $b^\uparrow$ for $b \in \Gamma(F_{\mathsf{c}})$ and linear vector fields $X \in \Gamma(F_A)$. By the involutivity of F_A, we have $D_X b \in \Gamma(F_{\mathsf{c}})$ for all $b \in \Gamma(F_{\mathsf{c}})$ and linear vector fields $X \in \Gamma(F_A)$. Hence, by Lemma A.3, we get that F_{c} is invariant under the flow lines of linear vector fields with values in F_A. That is, using the fact that ϕ_t^X is a vector bundle homomorphism, we have

$$\left(\phi_t^X \circ \phi_s^{b^\uparrow}\right)(a_m) \in \phi_t^X(a_m + F_{\mathsf{c}}(m)) = \phi_t^X(a_m) + F_{\mathsf{c}}\left(\phi_t^{\bar{X}}(m)\right)$$

for all $a_m \in A$, $t \in \mathbb{R}$ where this makes sense and $s \in \mathbb{R}$. Since

$$\phi_s^{b^\uparrow} \circ \phi_t^X(a_m) \in \phi_t^X(a_m) + F_{\mathsf{c}}\left(\phi_t^{\bar{X}}(m)\right),$$

the proof is finished.

(3) Assume that $a_m \sim a_n$ for some elements $a_m, a_n \in A$. Then there exists, without loss of generality, one linear vector field $X \in \Gamma(F_A)$ over $\bar{X} \in \Gamma(F_M)$, an element $b_m \in F_{\mathsf{c}}(m)$ and $t \in \mathbb{R}$ such that $a_m = \phi_t^X(a_n) + b_m$. We have then immediately

$$m = q_A(a_m) = q_A\left(\phi_t^X(a_n) + b_m\right) = \left(q_A \circ \phi_t^X\right)(a_n) = \phi_t^{\bar{X}}(n),$$

which shows $m \sim_{F_M} n$.

(4) Assume first that a does not vanish on its domain of definition. Since a is ∇-parallel, we have $D_X a \in \Gamma(F_{\mathsf{c}})$ for all linear vector fields $X \in \Gamma(F_A)$, $X \sim_{q_A} \bar{X} \in \Gamma(F_M)$. By Lemma A.3, this yields

$$\phi_t^X(a(m)) \in a\left(\phi_t^{\bar{X}}(m)\right) + F_{\mathsf{c}}\left(\phi_t^{\bar{X}}(m)\right) \tag{6.21}$$

for all $t \in \mathbb{R}$ where this makes sense and consequently

$$\pi\left(a(m)\right) = \pi\left(a\left(\phi_t^{\bar{X}}(m)\right)\right).$$

Since F_M is spanned by projections $\bar{X}$ of linear vector fields $X \in \Gamma(F_A)$, this shows that a projects to the map $[a]$.

In general, we have $a = \sum_{i=1}^n f_i a_i$ on an open set U with non-vanishing ∇-parallel sections $a_1, \ldots, a_n$ of A such that $a_1, \ldots, a_r \in \Gamma(F_{\mathsf{c}})$ and functions $f_1, \ldots, f_n \in C^\infty(U)$ such that $f_{r+1}, \ldots, f_n$ are F_M-invariant. This yields using (6.21):

$$\begin{aligned}\phi_t^X(a(m)) = \phi_t^X\left(\sum_{i=1}^n f_i(m) a_i(m)\right) &= \sum_{i=1}^n f_i(m)\phi_t^X(a_i(m))\\ &\in \sum_{i=r+1}^n f_i\left(\phi_t^{\bar{X}}(m)\right)\phi_t^X(a_i(m)) + F_{\mathsf{c}}\left(\phi_t^{\bar{X}}(m)\right)\\ &= a\left(\phi_t^{\bar{X}}(m)\right) + F_{\mathsf{c}}\left(\phi_t^{\bar{X}}(m)\right)\end{aligned}$$

and we get the statement in the same manner as above.

We have

$$([q_A] \circ [a]) \circ \pi_M = [q_A] \circ \pi \circ a = \pi_M \circ q_A \circ a = \pi_M \circ \mathrm{Id}_M = \pi_M,$$

which shows the last claim since π_M is surjective.

□

Corollary 6.2. *Let $(A \to M, F_M, F_{\mathsf{c}}, \nabla)$ be an IM-foliation on a Lie algebroid A. Choose $\bar{a}_m$ and $\bar{a}_n$ in A/F_{c}.*

(1) *Then $\pi_{\mathsf{c}}(\bar{a}_m) = \pi_{\mathsf{c}}(\bar{a}_n)$ if and only if $\overline{a_m} \in A/F_{\mathsf{c}}$ is the ∇-parallel transport of $\overline{a_n}$ over a piecewise smooth path along the foliation defined by F_M on M.*
(2) *If ∇ has trivial holonomy, then $\pi_{\mathsf{c}}(\overline{a_m}) = \pi_{\mathsf{c}}(\overline{a'_m})$ if and only if $\overline{a_m} = \overline{a'_m}$.*

Proof. (1) Assume first that $\pi_{\mathsf{c}}(\bar{a}_m) = \pi_{\mathsf{c}}(\bar{a}_n)$. Then there exists without loss of generality one linear vector field $X \in \Gamma(F_A)$ over $\bar{X} \in \Gamma(F_M)$ and $t \in \mathbb{R}$ such that $\bar{a}_m = \overline{\phi_t^X(a_n)}$.

Consider the curve $a : [0, t] \to A/F_{\mathsf{c}}$ over $c := \phi_\cdot^{\bar{X}}(n)$ defined by

$$a(\tau) = \overline{\phi_\tau^X(a_n)}$$

for $\tau \in [0, t]$. For each $\tau \in [0, t]$, we find $\varepsilon_\tau > 0$ and a parallel section a^τ of A such that $\overline{a^\tau(c(\tau))} = a(\tau)$. As in the proof of Proposition 6.1, 4), we get then that $\overline{a^\tau(c(s))} = a(s)$ for $s \in [\tau - \varepsilon_\tau, \tau + \varepsilon_\tau]$. This yields $\nabla_{\bar{X}(c(\tau))} a = 0$ for all τ.

Conversely, assume that $\overline{a_m} \in A/F_{\mathsf{c}}$ is the ∇-parallel transport of $\overline{a_n}$ over a piecewise smooth path along a path lying in the leaf of F_M through n. Without loss of generality, this path is a segment of a flow curve of a vector field $\bar{X} \in \Gamma(F_M)$, $m = \phi_t^{\bar{X}}(n)$ for some $t \in \mathbb{R}$, and there exists a ∇-parallel section a of A such that $\overline{a(m)} = \overline{a_m}$ and $\overline{a(n)} = \overline{a_n}$. Choose any linear vector field $X \in \Gamma(F_A)$ over $\bar{X}$. Then we get as in the proof of Proposition 6.1, 4) that

$$\overline{a(m)} = \overline{a\left(\phi_t^{\bar{X}}(n)\right)} = \overline{\phi_t^X(a(n))}$$

and hence $a_m \sim a_n$ by Proposition 6.1, 2).

(2) This is immediate since here, parallel transport does not depend on the path along the leaf of F_M through m.

□

Using Proposition 6.1, we show that if M/F_M is a smooth manifold and the projection is a submersion, then the quotient A/F_A is a vector bundle over M/F_M.

Proposition 6.3. *Let $(A \to M, F_M, F_{\mathsf{c}}, \nabla)$ be an IM-foliation on a Lie algebroid A. Assume that M/F_M is a smooth manifold such that the projection is a submersion, and that the connection ∇ has no holonomy. The quotient space A/F_A inherits a vector bundle structure over $\bar{M}$ such that the projection (π, π_M) is a vector bundle homomorphism.*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "q_A"'] & A/F_A \arrow[d, "{[q_A]}"] \\ M \arrow[r, "\pi_M"'] & M/F_M \end{tikzcd}$$

Proof. Choose a local frame for A/F_{c} of ∇-parallel sections $\bar{a}_1, \ldots, \bar{a}_k$ defined on $U \subseteq M$, $k = \operatorname{rank}(A/F_{\mathsf{c}})$. Write $q_{A/F_{\mathsf{c}}} : A/F_{\mathsf{c}} \to M$ for the vector bundle projection and consider the local trivialization of A/F_{c}:

$$\begin{array}{rccc} \Phi : & q_{A/F_{\mathsf{c}}}^{-1}(U) & \to & U \times \mathbb{R}^k \\ & \overline{b_m} & \mapsto & (m, \xi_1(b_m), \ldots, \xi_k(b_m)), \end{array}$$

where $b_m = \sum_{i=1}^k \xi_i(b_m) a_i(m) + c_m$ with some $c_m \in F_{\mathsf{c}}(m)$.

By Corollary 6.2, we find that for $\bar{b}_m, \bar{b}_n \in q_{A/F_{\mathsf{c}}}^{-1}(U)$, the equality

$$\pi_{\mathsf{c}}\left(\overline{b_m}\right) = \pi_{\mathsf{c}}\left(\overline{b_n}\right)$$

implies

$$\xi_i(b_m) = \xi_i(b_n) \text{ for } i = 1, \ldots, k,$$

since $\bar{b}_m$ is the parallel transport of $\bar{b}_n$ along any path in the leaf of F_M through m and n, and so in particular along a path in U. That is, the map Φ factors to a well-defined map

$$[\Phi] : [q_A]^{-1}(\bar{U}) \to \bar{U} \times \mathbb{R}^k$$

such that

$$[\Phi] \circ \pi_{\mathsf{c}} = (\pi_M, \mathrm{Id}_{\mathbb{R}^k}) \circ \Phi.$$

The map $[\Phi]$ is the projection to A/F_A of the "well chosen" local trivialization $q_{A/F_{\mathsf{c}}} \times \xi_1 \times \ldots \times \xi_k$ of A/F_{c}. Since by Proposition 2.4, we can cover A/F_{c} by this type of F_A-invariant trivializations, we find that we can construct trivializations for A/F_A, which is hence shown to be a vector bundle over $\bar{M}$. $\square$

Remark 6.4. Corollary 6.2 implies that the quotient space $\pi_{\mathsf{c}} : A/F_{\mathsf{c}} \to A/F_A$ is the quotient by the equivalence relation given by parallel transport. Constructions like this were made in [34], see also [16]. This idea will be used (implicitly) in the proofs of the following statements.

Note that this shows also that the data $(A, F_M, F_{\mathsf{c}}, \nabla)$ is an infinitesimal version of the ideal systems as in [21], and the methods of construction of the quotient algebroid are similar in this sense.

By construction, we have also the following vector bundle homomorphism

$$\begin{tikzcd} A/F_{\mathsf{c}} \arrow[r, "\pi_{\mathsf{c}}"] \arrow[d, "q_{A/F_{\mathsf{c}}}"'] & A/F_A \arrow[d, "{[q_A]}"] \\ M \arrow[r, "\pi_M"'] & M/F_M \end{tikzcd}$$

which is an isomorphism in every fiber, and we find that for each local section $[a]$ of A/F_A defined on $\bar{U} \subseteq \bar{M}$, there exists a ∇-parallel section a of A defined on $\pi_M^{-1}(\bar{U})$ such that $\pi \circ a = [a] \circ \pi_M$.

Proposition 6.5. *Let $(A, F_M, F_{\mathsf{c}}, \nabla)$ be an IM-foliation and assume that the quotient space $\bar{M} = M/F_M$ is a smooth manifold and ∇ has trivial holonomy. Then there is an induced map $[\rho] : A/F_A \to T\bar{M}$ such that*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\pi"'] & TM \arrow[d, "T\pi_M"] \\ A/F_A \arrow[r, "{[\rho]}"'] & T\bar{M} \end{tikzcd}$$

commutes.

If $a \in \Gamma(A)$ is ∇-parallel, then $\rho(a) \in \mathfrak{X}(M)$ is ∇^{F_M}-parallel and π_M-related to $[\rho][a] \in \mathfrak{X}(\bar{M})$.

Proof. Define $[\rho] : A/F_A \to T(M/F_M)$ by

$$[\rho]([a_m]) = T_m\pi_M(\rho(a_m)) \in T_{[m]}(M/F_M) \simeq T_mM/F_M(m).$$

To see that $[\rho]$ is well-defined, recall first that $\rho(F_{\mathsf{c}}) \subseteq F_M$. If $[a_m] = [a_n]$, then $a_m = b_m + \phi_t^X(a_n)$ for some linear section $X \in \Gamma(F_A)$ over $\bar{X} \in \Gamma(F_M)$, $t \in \mathbb{R}$ and $b_m \in F_{\mathsf{c}}(m)$. Consider the curve $a : [0, t] \to A/F_{\mathsf{c}}$ over $c := \phi_\cdot^{\bar{X}}(n)$ defined by

$$a(\tau) = \overline{\phi_\tau^X(a_n)}$$

for $\tau \in [0,t]$. Then $\nabla_{\bar{X}(c(\tau))} a = 0$ for all τ. For each $\tau \in [0,t]$, we find $\varepsilon_\tau > 0$ and a parallel section a^τ of A such that $a^\tau(c(s)) = a(s)$ for $s \in [\tau - \varepsilon_\tau, \tau + \varepsilon_\tau]$. Then, $\rho \circ a^\tau$ is ∇^{F_M}-parallel, and we get by Lemma 2.5 that $(\rho \circ a^\tau) \sim_{\pi_M} Y^\tau$ for some $Y^\tau \in \mathfrak{X}(\bar{M})$.

Since $[c(\tau - \varepsilon_\tau)] = [c(s)]$ for all $s \in [\tau - \varepsilon_\tau, \tau + \varepsilon_\tau]$, we have then

$$\begin{aligned} T_{c(\tau-\varepsilon_\tau)}\pi_M(\rho(a(\tau - \varepsilon_\tau))) &= T_{c(\tau-\varepsilon_\tau)}\pi_M(\rho(a^\tau(c(\tau - \varepsilon_\tau)))) \\ &= Y^\tau([c(\tau - \varepsilon_\tau)]) = Y^\tau([c(s)]) = T_{c(s)}\pi_M(\rho(a(s))) \end{aligned}$$

for all $s \in (\tau - \varepsilon_\tau, \tau + \varepsilon_\tau)$. Since $[0,t]$ is covered by intervals like this, we get $T_m\pi_M(\rho(a_m)) = T_m\pi_M(\rho(a(0))) = T_n\pi_M(\rho(a_n))$, which shows that $[\rho]$ is well-defined. □

Now we will define a Lie bracket on the space of sections of A/F_A. For $[a],[b] \in \Gamma(A/F_A)$, choose ∇-parallel sections $a, b \in \Gamma(A)$ such that $[a] \circ \pi_M = \pi \circ a$ and $[b] \circ \pi_M = \pi \circ b$. Then $[a,b]$ is ∇-parallel by the properties of ∇ and we can define

$$\Big[[a],[b]\Big]_{A/F_A} \in \Gamma(A/F_A)$$

by

$$\Big[[a],[b]\Big]_{A/F_A} \circ \pi_M = \pi \circ [a,b].$$

By the properties of ∇, this definition does not depend on the choice of the ∇-parallel sections (which can be made up to sections of F_{c}), and by definition and with Proposition 6.5, we get

$$[\rho]\left(\Big[[a],[b]\Big]_{A/F_A}\right) = \Big[[\rho][a],[\rho][b]\Big]_{T\bar{M}},$$

where the bracket on the right-hand side is the Lie bracket on vector fields on $\bar{M}$.

We can now complete the proof of the following theorem.

Theorem 6.6. *Let $(A, F_M, F_{\mathsf{c}}, \nabla)$ be an IM-foliation on a Lie algebroid A. Assume that $\bar{M} = M/F_M$ is a smooth manifold and ∇ has trivial holonomy. Then the triple $(A/F_A, [\rho], [\cdot\,,\cdot]_{A/F_A})$ is a Lie algebroid over $\bar{M}$ such that the projection (π, π_M) is a Lie algebroid morphism.*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "q_A"'] & A/F_A \arrow[d, "{[q_A]}"] \\ M \arrow[r, "\pi"'] & M/F_M \end{tikzcd}$$

Proof. The Jacobi identity follows immediately from the properties of the Lie bracket $[\cdot\,,\cdot]$ on $\Gamma_M(A)$ and the definition of $[\cdot\,,\cdot]_{A/F_A}$. For the Leibniz identity choose $[a],[b] \in \Gamma(A/F_A)$ corresponding to ∇-parallel sections $a, b \in \Gamma_M(A)$, and $f \in C^\infty(\bar{M})$. We have then $\pi_M^* f \in C^\infty(M)^{F_M}$ and $f \cdot [b]$ corresponds to the ∇-parallel section $(\pi_M^* f) \cdot b$ of A (see Lemma 2.5). We have hence

$$\begin{aligned} [[a], f \cdot [b]]_{A/F_A} \circ \pi_M &= \pi \circ [a, (\pi_M^* f) \cdot b] \\ &= \pi \circ (\pi_M^* f \cdot [a,b] + \rho(a)(\pi_M^* f) \cdot b) \\ &= \pi \circ (\pi_M^* f \cdot [a,b] + \pi_M^*([\rho][a](f)) \cdot b) \\ &= \big(f \cdot [[a],[b]]_{A/F_A} + [\rho][a](f) \cdot [b]\big) \circ \pi_M, \end{aligned}$$

where we have used Proposition 6.5 in the third equality.

The fact that (π, π_M) is compatible with the Lie algebroid brackets is immediate by construction and the compatibility of the anchor maps is given by the definition of $[\rho]$. □

Assume now that $F_G \subseteq TG$ is a multiplicative foliation on $G \rightrightarrows M$ such that the leaf space G/F_G is a Lie groupoid over the leaf space M/F_M (recall that there are topological conditions for this to be true [15]). The multiplicative foliation F_G determines an IM-foliation $(A, F_M, F_{\mathsf{c}}, \nabla)$ and a Lie algebroid $(A/F_A, [\rho], [\cdot,\cdot]_{A/F_A})$ as in the preceding theorem. We conclude this subsection with the comparison of this Lie algebroid with the Lie algebroid of the leaf space $G/F_G \rightrightarrows M/F_M$.

Theorem 6.7. *Let $(A, F_M, F_{\mathsf{c}}, \nabla)$ be an IM-foliation on A. Assume that A integrates to a Lie groupoid $G \rightrightarrows M$, and F_A to a multiplicative foliation F_G on G. If G/F_G and M/F_M are smooth manifold, ∇ has trivial holonomy and F_G is such that $G/F_G \rightrightarrows M/F_M$ is a Lie groupoid, then we have*

$$A(G/F_G) = A/F_A,$$

where A/F_A is equipped with the Lie algebroid structure in the previous theorem.

Remark 6.8. It would be interesting to study the relation between the trivial holonomy property of ∇ and the condition of F_G for $G/F_G \rightrightarrows M/F_M$ to be a Lie groupoid.

Proof of Theorem 6.7. Let $\pi_G : G \to G/F_G$ be the projection, and $[\mathsf{s}], [\mathsf{t}]$ the source and target maps of $G/F_G \rightrightarrows M/F_M$. Recall from Theorem 3.6 that a section $a \in \Gamma(A)$ is ∇-parallel if and only if $[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G)$ and the vector field a^r is then π_G-related to a vector field $\overline{a^r} \in \mathfrak{X}(G/F_G)$. We have

$$T[\mathsf{s}] \circ \overline{a^r} \circ \pi_G = T[\mathsf{s}] \circ T\pi_G \circ a^r = T\pi_G(T\mathsf{s} \circ a^r) = 0,$$

which shows that $\overline{a^r}$ is tangent to the $[\mathsf{s}]$-fibers. By Lemma 3.18 in [15], we get

$$\overline{a^r}([g]) = T_g\pi_G(a^r(g)) = T_g\pi_G(a(\mathsf{t}(g)) \star 0_g) = T_{\mathsf{t}(g)}\pi_G(a(\mathsf{t}(g))) \star 0_{[g]} = \overline{a^r}([\mathsf{t}][g]) \star 0_{[g]},$$

which shows that $\overline{a^r} = \tilde{a}^r$ for $\tilde{a} := \overline{a^r}|_{M/F_M} \in \Gamma(A(G/F_G))$.

Since (π_G, π_M) is a Lie groupoid morphism

$$\begin{array}{ccc} G & \xrightarrow{\pi_G} & G/F_G \\ {\scriptstyle \mathsf{t}}\downarrow\downarrow{\scriptstyle \mathsf{s}} & & {\scriptstyle [\mathsf{t}]}\downarrow\downarrow{\scriptstyle [\mathsf{s}]} \\ M & \xrightarrow[\pi_M]{} & M/F_M \end{array}$$

the map $A(\pi_G) = T\pi_G|_A$

$$\begin{array}{ccc} A & \xrightarrow{A(\pi_G)} & A(G/F_G) \\ \downarrow & & \downarrow \\ M & \xrightarrow[\pi_M]{} & M/F_M \end{array}$$

is a Lie algebroid morphism and $A(\pi_G)(b_m) = 0$ for all $b_m \in F_{\mathsf{c}}$. For any ∇-parallel section $a \in \Gamma(A)$ we have $a^r \sim_{\pi_G} \tilde{a}^r$ and hence

$$A(\pi_G) \circ a = T\pi_G a = \tilde{a} \circ \pi_M. \tag{6.22}$$

Define the map

$$\begin{array}{ccc} A/F_A & \xrightarrow{\Psi} & A(G/F_G) \\ \downarrow & & \downarrow \\ \bar{M} & \xrightarrow[\mathrm{Id}_M]{} & M/F_M \end{array}$$

by

$$\Psi([a_m]) = A(\pi_G)(a_m)$$

for all $a_m \in A$. To see that this does not depend on the representative, note that $F_{\mathsf{c}} = \ker(A(\pi_G))$ and recall that $a_m \sim a_n$ if and only if $\bar{a}_m$ is the ∇-parallel transport of $\bar{a}_n$ along a path lying in the leaf through m of F_M (Corollary 6.2). Without loss of generality, there exists a ∇-parallel section $a \in \Gamma(A)$ such that $a(m) = a_m$ and $a(n) = a_n + b_n$ for some $b_n \in F_{\mathsf{c}}(n)$. Then, using (6.22), we get

$$\begin{aligned}\Psi([a_m]) &= A(\pi_G)(a_m) = (A(\pi_G) \circ a)(m) = (\tilde{a} \circ \pi_M)(m) \\ &= (\tilde{a} \circ \pi_M)(n) = A(\pi_G)(a_n) = \Psi([a_n]).\end{aligned}$$

Hence, Ψ is a well-defined vector bundle homomorphism over the identity on M/F_M. Furthermore, the considerations above show that for any section $[a]$ of A/F_A, and corresponding ∇-parallel section a of A, we get

$$\Psi \circ [a] = \tilde{a}.$$

The compatibility of the Lie algebroid brackets and anchors is then immediate by the construction of A/F_A, and the fact that $A(\pi_G)$ is a Lie algebroid morphism. □

7. Further remarks and applications

7.1. **Regular Dirac structures and the kernel of the associated presymplectic groupoids.** Let (M, D) be a Dirac manifold. Recall that $(\mathsf{D} \to M, \mathrm{pr}_{TM}, [\cdot\,,\cdot])$ is then a Lie algebroid, where $[\cdot\,,\cdot]$ is the Courant Dorfman bracket on sections of $TM \oplus T^*M$.

Assume that the characteristic distribution $F_M \subseteq TM$, defined by

$$F_M(m) = \{v_m \in T_mM \mid (v_m, 0_m) \in \mathsf{D}(m)\}$$

for all $m \in M$, is a subbundle of TM. The involutivity of F_M follows from the properties of the Dirac structure. Set $F_{\mathsf{c}} := F_M \oplus \{0\} \subseteq \mathsf{D}$. It is easy to check that F_{c} is a subalgebroid of D. Define

$$\begin{aligned}\nabla : \Gamma(F_M) \times \Gamma(\mathsf{D}/F_{\mathsf{c}}) &\to \Gamma(\mathsf{D}/F_{\mathsf{c}}) \\ \nabla_X(\bar{d}) &= \overline{[(X,0), d]}.\end{aligned}$$

This map is easily seen to be a well-defined, flat, partial F_M-connection on $\mathsf{D}/F_{\mathsf{c}}$, and the verification of the fact that $(\mathsf{D}, F_M, F_{\mathsf{c}}, \nabla)$ is an IM-foliation on the Lie algebroid $\mathsf{D} \to M$ is straightforward.

We show that if $\mathsf{D} \to M$ integrates to a presymplectic groupoid $(G \rightrightarrows M, \omega_G)$ [6, 4], then $(\mathsf{D}, F_M, F_{\mathsf{c}}, \nabla)$ integrates to the foliation $F_G = \ker \omega_G \subseteq TG$.

The map $\rho := \mathrm{pr}_{TM} : \mathsf{D} \to TM$ is the anchor of the Dirac structure D viewed as a Lie algebroid over M, and the map $\sigma := \mathrm{pr}_{T^*M} : \mathsf{D} \to T^*M$ defines an IM-2-form on the Lie algebroid D (see [6], [4]). Note that F_{c} is the kernel of σ and F_M is the kernel of $\sigma^t : TM \to \mathsf{D}^*$. The two-form $\Lambda := \sigma^*\omega_{\mathrm{can}} \in \Omega^2(\mathsf{D})$ is morphic in the sense that

$$\begin{tikzcd} T\mathsf{D} \arrow[r, "\Lambda^\sharp"] \arrow[d] & T^*\mathsf{D} \arrow[d] \\ TM \arrow[r, "-\sigma^t"'] & \mathsf{D}^* \end{tikzcd}$$

is a Lie algebroid morphism ([4]). See, for instance [21], for the Lie algebroid structure on $T^*\mathsf{D} \to \mathsf{D}^*$. If $\mathsf{D} \to M$ integrates to a presymplectic groupoid $(G \rightrightarrows M, \omega_G)$, the Lie algebroid $T^*\mathsf{D} \to \mathsf{D}^*$ is isomorphic to the Lie algebroid of the cotangent groupoid $T^*G \to \mathsf{D}^*$ and the map $\Lambda^\sharp$ integrates via the identifications $T\mathsf{D} \simeq A(TG)$ and $T^*\mathsf{D} \simeq A(T^*G)$ to the vector bundle map $\omega_G^\sharp$, that is a Lie groupoid morphism. See [4] for more details.

We show that the morphic foliation $F_{\mathsf{D}} \subseteq T\mathsf{D}$ defined by $(\mathsf{D}, F_M, F_{\mathsf{c}}, \nabla)$ as in Theorem 5.5 is equal to the kernel of $\Lambda^\sharp$.

Let n be the dimension of M and k the rank of F_M. Then D is spanned locally by frames of n parallel sections, the first k of them spanning F_{c}. If d is a parallel section of D, we have $\pounds_X d \in \Gamma(F_{\mathsf{c}})$ for all $X \in \Gamma(F_M)$, that is, $\pounds_X(\sigma(d)) = 0$ for all $X \in \Gamma(F_M)$. Since $\mathbf{i}_X\sigma(d) = 0$ for all $X \in \Gamma(F_M)$, this yields $\mathbf{i}_X\mathbf{d}(\sigma(d)) = 0$ for all $X \in \Gamma(F_M)$. Hence, using this type of frames, we find with formulas (4.57) and (4.58) in [4], that the kernel of $\Lambda^\sharp$ is spanned by the restriction to F_M of the linear sections defined by parallel sections of D, and by the restrictions to F_M of the core sections defined by sections of F_{c}. Hence, by construction, the foliation F_{D} is the kernel of $\Lambda^\sharp$.

Since the kernel of $\omega_G^\sharp$ is multiplicative with Lie algebroid equal to the kernel of $\Lambda^\sharp$, this yields $F_G = \ker \omega_G^\sharp$.

Note that if $F_M \subseteq TM$ is simple, then the leaf space M/F_M has a natural Poisson structure such that the projection $(M, \mathsf{D}) \to (M/F_M, \pi)$ is a forward Dirac map. Under a completeness condition and if $F_G \subseteq TG$ is also simple, we get a Lie groupoid $G/F_G \rightrightarrows M/F_M$, with a natural symplectic structure ω such that the projection $\pi_G : G \to G/F_G$ satisfies $\pi_G^*\omega = \omega_G$. It would be interesting to study the relation between the integrability of the Poisson manifold $(M/F_M, \pi)$ and the completeness conditions on F_G (see [15]) so that the quotient $(G/F_G \rightrightarrows M/F_M, \omega)$ is a symplectic groupoid.

7.2. **Foliated algebroids in the sense of Vaisman.** In [33], foliated Lie algebroids are defined as follows. A foliated Lie algebroid is a Lie algebroid $A \to M$ together with a subalgebroid B of A and an involutive subbundle $F_M \subseteq TM$ such that

(1) $\rho(B) \subseteq F_M$,
(2) A is locally spanned over $C^\infty(M)$ by *B-foliated cross sections*, i.e., sections a of A such that $[a, b] \in \Gamma(B)$ for all $b \in B$.

Recall our definition of IM-foliation on a Lie algebroid (Definition 5.1). Since the F_M-partial connection is flat, we get by Proposition 2.4 the existence of frames of parallel sections for A. By the properties of the connection, these are *F_{c}-foliated cross sections*. Since (1) is also satisfied by hypothesis, our IM-foliations are foliated Lie algebroids in the sense of Vaisman if we set $B := F_{\mathsf{c}}$.

The object that integrates the foliated algebroid in the sense of Vaisman is the right invariant image of B, which defines a foliation on G that is tangent to the s-fibers and invariant under left multiplication. This is exactly the intersection of our multiplicative subbundle $F_G \subseteq TG$ integrating $(A \to M, F_M, F_{\mathsf{c}}, \nabla)$ with $T^{\mathsf{s}}G$.

Appendix A. Invariance of bundles under flows

We prove here a result that is standard, but the proof of which is difficult to find in the literature.

Theorem A.1. *Let M be a smooth manifold and E be a subbundle of the direct sum vector bundle $\mathbb{T}M := TM \oplus T^*M$. Let $Z \in \mathfrak{X}(M)$ be a smooth vector field on M and denote its flow by ϕ_t. If*

$$\pounds_Z e \in \Gamma(E) \quad \textit{for all} \quad e \in \Gamma(E),$$

then

$$\phi_t^* e \in \Gamma(E) \quad \textit{for all} \quad e \in \Gamma(E) \quad \textit{and} \quad t \in \mathbb{R} \textit{ where this makes sense.}$$

Corollary A.2. *Let F be a subbundle of the tangent bundle TM of a smooth manifold M. Let $Z \in \mathfrak{X}(M)$ be a smooth vector field on M and denote its flow by ϕ_t. If*

$$[Z, \Gamma(F)] \subseteq \Gamma(F),$$

then

$$T_m\phi_t F(m) = F(\phi_t(m))$$

for all $m \in M$ and t where this makes sense.

Proof. Choose $X \in \Gamma(F)$ and $m \in M$. Then, by Theorem A.1, we have $T\phi_t \circ X \circ \phi_{-t} = \phi_{-t}^* X \in \Gamma(F)$ for all t where this makes sense, and hence:

$$T_m\phi_t X(m) = \left(\phi_{-t}^* X\right)(\phi_t(m)) \in F(\phi_t(m)).$$

□

Proof of Theorem A.1. The subbundle E of $\mathbb{T}M$ is an embedded submanifold of $\mathbb{T}M$. For each section σ of $\mathbb{T}M$, the smooth function $l_\sigma : \mathbb{T}M \to \mathbb{R}$ is defined by

$$l_\sigma(v, \alpha) = \langle \sigma(p(v, \alpha)), (v, \alpha) \rangle$$

for all $(v, \alpha) \in \mathbb{T}M$, where $p : \mathbb{T}M \to M$ is the projection. For all $e \in E$, the tangent space T_eE of the submanifold E of $\mathbb{T}M$ is equal to

$$\ker \left\{ \mathbf{d}_e l_\sigma \mid \sigma \in \Gamma\left(E^\perp\right) \right\}.$$

Consider the complete lift $\tilde{Z}$ to $\mathbb{T}M$ of Z, i.e., the vector field $\tilde{Z} \in \mathfrak{X}(\mathbb{T}M)$ defined by

$$\tilde{Z}(l_\sigma) = l_{\pounds_Z \sigma} \qquad \text{and} \qquad \tilde{Z}(p^* f) = p^*(Z(f))$$

for all $\sigma \in \Gamma(\mathbb{T}M)$ and $f \in C^\infty(M)$ (see [21]).

Choose $e \in E$ and $\sigma \in \Gamma\left(E^\perp\right)$. Then we have $\pounds_Z \sigma \in \Gamma\left(E^\perp\right)$ since for all $\tau \in \Gamma(E)$:

$$\langle \pounds_Z \sigma, \tau \rangle = Z\left(\langle \sigma, \tau \rangle\right) - \langle \sigma, \pounds_Z \tau \rangle = 0.$$

This leads to

$$(\mathbf{d}_e l_\sigma)(\tilde{Z}(e)) = \left(\tilde{Z}(l_\sigma)\right)(e) = l_{\pounds_Z \sigma}(e) = 0.$$

Hence, the vector field $\tilde{Z}$ is tangent to E on E. As a consequence, its flow curves starting at points of e remain in the submanifold E.

It is easy to check that the flow Φ_t of the vector field $\tilde{Z}$ is equal to $(T\phi_t, (\phi_{-t})^*)$, i.e.,

$$\Phi_t(v_m, \alpha_m) = (T_m\phi_t(v_m), \alpha_m \circ T_{\phi_t(m)}\phi_{-t})$$

for all $(v_m, \alpha_m) \in \mathbb{T}M(m)$. Choose a section $(X, \alpha) \in \Gamma(E)$ and a point $m \in M$. We find

$$(\phi_t^*(X, \alpha))(m) = \left(T_{\phi_t(m)}\phi_{-t} X(\phi_t(m)), \alpha_{\phi_t(m)} \circ T_m\phi_t\right) = \Phi_{-t}\left((X, \alpha)(\phi_t(m))\right) \in E(m)$$

since $(X, \alpha)(\phi_t(m)) \in E(\phi_t(m))$. Thus, we have shown that $\phi_t^*(X, \alpha)$ is a section of E. □

Assume now that $q_A : A \to M$ is a vector bundle, and consider a linear vector field X on A, i.e., the map $X : A \to TA$ is a vector bundle homomorphism over $\bar{X} : M \to TM$ such that $X \sim_{q_A} \bar{X}$. Let $\phi_\cdot^X$ be the flow of X and $\phi_\cdot^{\bar{X}}$ the flow of $\bar{X}$. Then $\phi_t^X : A \to A$ is a vector bundle homomorphism over $\phi_t^{\bar{X}}$ for all $t \in \mathbb{R}$ where this is defined.

Recall that for any $a \in \Gamma(A)$, the section $D_X a \in \Gamma(A)$ is defined by

$$(D_X a)(m) = \left.\frac{d}{dt}\right|_{t=0} \phi_{-t}^X(a(\phi_t^{\bar{X}}(m)))$$

for all $m \in M$. In the same manner, if $\varphi \in \Gamma(A^*)$, we can define

$$(D_X \varphi)(m) = \left.\frac{d}{dt}\right|_{t=0} (\phi_t^X)^*(\varphi(\phi_t^{\bar{X}}(m)))$$

for all $m \in M$. We have then $\varphi(a) \in C^\infty(M)$, and

$$\bar{X}(m)(\varphi(a)) = \varphi(D_X a)(m) + (D_X \varphi)(a)(m). \tag{1.23}$$

We can hence show the following lemma.

Lemma A.3. *Let A be a vector bundle and $B \subseteq A$ a subbundle.*

(1) *If $(X, \bar{X})$ is a linear vector field on A such that*
$$D_X b \in \Gamma(B)$$
for all $b \in \Gamma(B)$, then $\phi_t^X(b_m) \in B\left(\phi_t^{\bar{X}}(m)\right)$ for all $b_m \in B_m$.

(2) *Assume furthermore that $a \in \Gamma(A)$ is such that $a(m)$ is linearly independent to $B(m)$ for all m in $\mathrm{Dom}(a)$ and*
$$D_X a \in \Gamma(B).$$
Then
$$\phi_t^X(a(m)) \in a\left(\phi_t^{\bar{X}}(m)\right) + B\left(\phi_t^{\bar{X}}(m)\right)$$
for all $m \in U$ and $t \in \mathbb{R}$ where this makes sense.

Proof. (1) We check that the vector field X is tangent to B on points in B. Let $\varphi \in \Gamma(A^*)$ be a section that vanishes on B, i.e., $\varphi_m(b_m) = 0$ for all $b_m \in B$. Let $l_\varphi \in C^\infty(A)$ be the linear function defined by φ. By (1.23), we have then $D_X\varphi \in \Gamma(B^\circ)$. Choose $b_m \in B$. We have then
$$\begin{aligned}\mathbf{d}_{b_m} l_\varphi(X(b_m)) &= \left.\frac{d}{dt}\right|_{t=0} l_\varphi(\phi_t^X(b_m)) = \left.\frac{d}{dt}\right|_{t=0} \varphi_{\phi_t^{\bar{X}}(m)}(\phi_t^X(b_m))\\ &= (D_X\varphi)(b_m) = 0.\end{aligned}$$
Thus, X is tangent to B on B and the flow of X preserves B.

(2) Assume now that $(b_1, \ldots, b_k)$ is a local frame for B on an open set $U \subseteq M$. Complete this frame to a local frame $(b_1, \ldots, b_n)$ for A defined on an open U such that $b_{k+1} := a \in \Gamma(A)$. Let $\varphi_1, \ldots, \varphi_n$ be a frame for A^* that is dual to $(b_1, \ldots, b_n)$, i.e., such that $(\varphi_{k+1}, \ldots, \varphi_n)$ is a frame for B° and $\varphi_{k+1}(a) = 1$. Then, the closed submanifold C of $A|_U$ defined by $C(m) = a(m) + B(m)$ is the level set with value $(1, 0, \ldots, 0)$ of the function
$$(l_{\varphi_{k+1}}, \ldots, l_{\varphi_n}) : A|_U \to \mathbb{R}^{n-k}.$$
Since $D_X a \in \Gamma(B)$ for the linear vector field $(X, \bar{X})$ on A, we get
$$0 = \bar{X}(\varphi_i(a)) = \varphi_i(D_X a) + D_X\varphi_i(a) = 0 + D_X\varphi_i(a)$$
for $i = k+1, \ldots, n$ and this yields as before for all $b_m \in B$:
$$\begin{aligned}\mathbf{d}_{a(m)+b_m} l_{\varphi_i}(X(a(m)+b_m)) &= \left.\frac{d}{dt}\right|_{t=0} l_{\varphi_i}(\phi_t^X(a(m)+b_m))\\ &= (D_X\varphi_i)(a(m)+b_m) = 0.\end{aligned}$$
Hence, X is tangent to C on points of C. That is, the flow of X preserves C and the proof is finished.

□

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